Repeated Bargaining With Reference-Dependent Preferences*

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Abstract

We consider a two-player bargaining model in which one or both players have reference-dependent preferences, but are otherwise perfectly rational. Our behavioural assumption is that players with reference-dependent preferences prefer impasse to consuming strictly less than their current reference points. Reference points adjust each period according to some exogenously specified law of motion. When reference points do not adjust following disagreement, we show that disagreement does not arise in equilibrium, but they do influence the division of the pie. In contrast, when reference points adjust downwards following disagreement, disagreements arise and players may try to manipulate the reference point of their opponent. When reference points adjust downwards following a rejection, for a particular Markov equilibrium, we show that the set of feasible allocations can be divided into agreement and disagreement regions. In particular, there are thresholds such that if one (or more) player's reference point is above the threshold, disagreement necessarily arises.

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1 Introduction

There is a large experimental literature on bargaining which explores the dynamics of bargaining and the nature of agreements and disagreements. Two conclusions stand out: first, in

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most bargaining situations, bargaining is a gradual process, concluding only after a number of periods. Second, there is a strictly positive probability of disagreement. In an interesting experimental work, Gächter and Riedl [4] argue that the presence of reference points can have a strong impact on the bargaining outcome. In particular, they show that many subjects appear to have reference points, or in their terminology, *moral property rights* and that, when the pie over which subjects bargain makes these reference points infeasible, disagreement is a much more likely outcome. This body of experimental evidence has led to some recent theoretical work on reference points in bargaining. In particular, both Compte and Jehiel [3] and Li [7] consider players with reference-dependent preferences and generate a gradual bargaining process. In the former paper, a high offer is not made since a player may then reject it in order to increase her reference point and, therefore, improve her bargaining position. In the latter, offers become increasingly generous until an agreement is finally reached.

Both of these models concern themselves only with a single bargaining period. Our interest lies in examining repeated bargaining situations. Many real world situations seem better described by repeated interaction. For example, a union and a firm sign a contract with only a finite duration. Eventually then, a new contract must be negotiated. It seems likely that the prospect of future negotiations are considered in present negotiations. Surely matters are substantially different if the firm knows that by acquiescing to the union's demands today it will create a new status quo for the next negotiation.

In a similar vein, governments must decide upon social benefits and tax policy — on this front there is much evidence that reference points play an important role in the discussion.[1] For example, despite being highly controversial in the 1960s, Medicare in the United States is now seen, by many Democrats and Republicans alike, as being largely "untouchable". Similarly, with the recent healthcare reform debate in the United States, many feared that the reforms would lead to another "entitlement" for citizens that would be extremely difficult to claw back should circumstances require it in the future. Similarly, tax cuts, once given to a large portion of the voting population, are extremely difficult to rescind.

Below, we parameterise one way in which reference points may enter into a decision maker's utility function. Our main behavioural assumption is that a decision maker with such preferences will not accept anything below her current reference point. Our parameterisation assumes that reference points adjust upwards instantaneously but only adjust downwards gradually (*i.e.,* by a factor $\alpha \in (0, 1)$) or not all (*i.e.,* $\alpha = 1$). While other parameterisations of reference points may be possible, the particular rule that we use is tractable enough to allow us to prove the existence of Markov equilibria.

When reference points adjust endogenously (and are free to adjust downwards following

[1] See, for example, Romer [9] and Boeri *et al* [1].

a disagreement), players may try to strategically manipulate each other's reference points to their advantage. In particular, it seems plausible that a situation akin to *breaking the union* may occur in which one player, the firm, constantly rejects offers in order to lower the reference point of the union in order to gain some advantage in future negotiations. It is then interesting to ask the following: what are the stable outcomes of such a bargaining process?

The present work shares some similarities to that of Ghosh [5]. In his model, N agents are randomly paired in each period and must decide how to divide a pie of size 1. If the sum of the matched players' aspirations is less than 1 then the pie is divided, while if the aspirations sum to more than one, each player gets nothing. Once the payoffs in the current period are realised, the aspirations of each player evolve — upwards if the sum of aspirations was strictly less than one and downwards if it was strictly greater than one. Ghosh [5] shows that the only stochastically stable outcome is when each player has an aspiration level of exactly half the pie. As in Ghosh's model our players have aspiration levels. However, we assume that the reference level is the only way in which an agent may be irrational; that is, all agents seek to maximise the discounted sum of their utility, while at the same time being fully aware of the role of reference points. Thus there is much room for strategic behaviour when we turn to the bargaining problem in the sections below.

In the next section we describe our main behavioural assumptions; in particular, we highlight the role of reference points and how they are updated based upon observed outcomes. Our main analysis begins in Section 3 in which we consider the case in which one player is fully rational, while the other player's utility function is modified by the presence of a reference point, and a particular rule for updating reference points in each period. We will call the player with reference-depedent preferences the *behavioural player*. When reference points do not adjust downward following a disagreement, we demonstrate the existence of a Markov equilibrium in which the reference point of the behavioural player gradually increases over time, before (in the limit) exhausting the entire pie. We also show that this equilibrium may be used to construct other equilibria in which the reference point of the behavioural player gradually increases to a limit strictly less than the entire pie. Such equilibria are supported by reversion to the aforementioned Markov equilibrium. On the other hand, when reference points adjust downwards following a disagreement, we show that there does not exist a Markov equilibrium in which the behavioural player eventually exhausts the entire pie. Instead, we construct a Markov equilibrium in which there is some reference point, $\bar{r}$, above which disagreement necessarily arises and below which reference points gradually increase, limiting to $\bar{r}$. Thus $\bar{r}$ represents the long-run proportion of the pie that the behavioural player can expect to obtain. Our analysis shows that this share increases as reference points adjust downwards more slowly. In contrast, the share decreases the more patient the players are.

In Section 4 we analyse the case in which both players are behavioural players. Regardless of whether reference points can adjust downwards following a disagreement, we show that many of the properties of Rubinstein [11] continue to hold. In the case of downwards fixed reference points, agreement is reached immediately and, for sufficiently small reference points, the outcome coincides with Rubinstein [11]. When reference points do adjust downwards following disagreement, we show that the bargaining space can generally be divided into regions of agreement and disagreement. Disagreements occur when reference points are initially very unequal and are useful because by forcing disagreement, the inequality in reference points is diminished, allowing the *weaker* player to increase his or her share. We also characterise the bargaining region for a particular Markov equilibrium which satisfies a number of desirable properties. We show that the acceptance region is shrinking as players become more patient and as reference points adjust downwards more rapidly following disagreement. Just as when reference points are downwards fixed, we also show that when players are sufficiently patient, the final stable outcome is very similar to that of Rubinstein [11].

In Section 5 we discuss the related literature in more detail, with particular emphasis on Compte and Jehiel [3], while in Section 6 we provide some concluding remarks.

2 Preliminaries

There are two players who, in each period $t \in \mathbb{N}$, bargain over a pie of size 1. In each period, one player is randomly selected to make a take-it-or-leave it offer. If the offer is accepted, then the pie in that period is split according to the agreed upon division, while if the offer is rejected, both players get 0 in time t. In either case, the game moves on to period $t+1$ where a new proposer is randomly chosen. For ease of simplicity, we assume that each player is equally likely to be chosen to propose in any period.

The above set-up is standard. Our point of departure is the preferences of one, or both, of the players. In particular, we assume that players may have a particular form of reference-dependent utility. We will call such players, *behavioural* players. Indeed, preferences of a behavioural player take the following form: Given a reference point $r \geq 0$,

$$u(c,r) = \begin{cases} c & \text{if } c \geq r \\ 0 & \text{if } c = 0 \\ -\infty & \text{if } 0 < c < r \end{cases} \tag{1}$$

This utility function captures the assumption that the decision maker strictly prefers to consume nothing, say because she has an outside option which guarantees her zero utility, than to consume any amount strictly less than her reference point, or because of some irrational notion that consumption today should be at least as large as consumption yesterday.[2] For example, a decision maker who was raised on Dom Perignon champagne may, quite reasonably, prefer not to drink any champagne at all rather than drink Korbel sparkling wine.[3]

Note that our specification of the utility function is different from that in the habit formation literature.[4] Under the assumption of habit formation, utility depends positively on the rate of consumption growth. In the champagne example, this amounts to saying that the utility from one's first ever glass of Dom Perignon is higher than the utility of the n^{th} glass. Since we focus on the case in which players repeatedly bargain over a pie of constant size, we feel that preferences based on habit formation are less natural and, therefore, prefer the specification given by (1).

Given the utility function specified above, we must now specify how reference points evolve. We will assume the following exogenous process:

$$r_{t+1} = \begin{cases} c_t & \text{if } c_t \geq r_t \\ \alpha r_t & \text{if } c_t = 0 \end{cases} \tag{2}$$

where $\alpha \in [0,1]$ parameterises the speed of downward adjustment of reference points following periods of zero consumption. If $\alpha = 1$, we say that referenced points are *downwards unadjustable* (or, interchangeably, *downwards fixed*), while when $\alpha < 1$, we say that reference points are *downwards adjustable*. Notice that reference points adjust upwards very rapidly to the previous period's level of consumption but only adjust downwards gradually.

Finally, we assume that both players have a common discount factor $\delta \in (0,1)$ and that both players, whether behavioural or not, maximise the present value of expected utility. Also, let $\Delta = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1\}$. We will call Δ the bargaining space (or, interchangeably, the bargaining region).

We make no assumption on where initial reference points come from and instead characterise the solution for all possible initial reference points. In some instances it may be reasonable to assume that reference points start at 0 for both players, while in others, strictly positive, and possibly asymmetric, initial reference points may be more appropriate. For example, when entering into a new situation with another player, one might reasonably be expected to transfer his/her expectations of fairness derived from previous encounters.

[2]For example, Loewenstein and Shicerman [8] showed that many subjects had strongly preferred an increasing wage profile, even when such a profile did not maximise the present value of income.

[3]This is really just an extreme version of the *kinked* utility function. In particular, the slope of the utility function below the kink is $-\infty$. One could relax this assumption but only at the cost of significantly less tractability.

[4]For an example applied to growth models, see Carroll *et al* [2].

Remark 1. *Though extreme, the updating process given in (2) captures all of the features we feel important in examining a model of bargaining with reference points. At the cost of tractability, one could conceive of many other, perhaps more realistic, laws of motion for reference points. For example, one could consider adding inertia in both directions in the following way:*

$$r_{t+1} = \alpha r_t + (1 - \alpha)c_t \tag{3}$$

The speed of adjustment is then given by α. We do not consider this law of motion further, except to note that adding upward inertia lowers the cost of increasing consumption. Therefore, all else equal, we would expect higher levels of consumption with upward inertia than with instantaneous upward adjustment. Indeed, if $\alpha = 1$ and $r = 0$, our decision maker is perfectly rational and would freely consume the entire pie in every period.

3 Repeated Bargaining When One Player Has Reference-Dependent Preferences

In this section we examine the equilibria of the game in which a fully rational player repeated bargains over fixed pie with a behavioural player with reference-dependent preferences in the manner described above.

3.1 Downwards Unadjustable Reference Points

Suppose that one player, without loss of generality, player 1 is fully rational, while player 2 has preferences given by (1) and reference points which are downwards unadjustable. We restrict attention to Markov strategies by both players. That is, if $r_t \in [0, 1]$ is the reference point of player 2 in period t, then the proposer's strategy is a function of r_t only and the responder's decision to accept or reject depends only upon r_t and the proposal. We also assume a stationary bargaining protocol in which each player is equally likely to be the proposal in any given period.

We first show that there exists a Markov equilibrium in which the reference point of the behavioural player approaches 1. That is, eventually the behavioural player consumes the entire pie, leaving the fully rational player with nothing. The equilibrium strategies have the following flavour. In each period t, if the rational player is selected to propose, she proposes r_t to the behavioural player and $1 - r_t$ for herself. On the other hand, if the behavioural player is selected to propose, he will propose $1 - f(r_t)$ to he rational player and $f(r_t) > r_t$ to himself, where $f(r_t)$ is determined as part of the equilibrium. In particular, $f(r)$ is chosen small enough such that the rational player is indifferent between accepting and rejecting.

Proposition 1. *There exists a Markov equilibrium such that, given reference point r_t in period t (i) the rational player proposes r_t to the behavioural player, who accepts any proposal $y \geq r_t$, and (ii) the behavioural player proposes $f(r_t) = r_t + \frac{2(1-r_t)(1-\delta)}{2-\delta}$ to himself and $1 - f(r_t)$ to the rational player, who accepts any proposal $z \geq 1 - f(r_t)$.*

Proof. See appendix. □

Although in the above equilibrium, the rational player must eventually cede the entire pie to the behavioural player, the outcome is actually fair in terms of the expected present value. In particular, the expected present value to both players at $r = 0$ is $\frac{1}{2}$. In the equilibrium, the rational player obtains a disproportionate piece of the pie in early periods and gradually cedes some of profits to the behavioural player, who eventually consumes the entire pie. Of course, if the initial reference point of the behavioural player is different from 0, then even this equilibrium is biased in favour of the behavioural player.

Of course, the above equilibrium is just one of many possible equilibria. We would like to obtain a better understanding of other equilibria which may exist. To do so, some terminology is in order. Given an equilibrium strategy profile σ, say that the state r is *absorbing* if, whenever the state is r, both players propose r when it is their turn to do so. Note that in the above proposition, there is a unique absorbing state at $r = 1$. As a preliminary result, it is fairly easy to see that the following holds:

Lemma 1. *In any Markov equilibrium, the set of absorbing states cannot contain any intervals.*

Proof. See appendix. □

We now provide an example showing that the rational player may, in fact, do very well bargaining against a player with downwards fixed reference points.

Example 1. *Modify the equilibrium strategies defined in Proposition 1 at $r = 0$. In particular, at $r = 0$, the modified strategy calls for both players to propose $y = 1$ to the rational player, for the rational player 1 to accept all offers such that player 2 gets 0 and to reject any offer such that player two gets a strictly positive amount. At $r = 0$, player 2 accepts any proposal. For all $r > 0$, the strategies are as in Proposition 1.*

We claim that this is an equilibrium provided that $\delta \geq \frac{2}{3}$. First, observe that player 2 is indifferent between all proposals he could make; therefore, it is weakly optimal to cede the entire pie to player 1. Second, player 2's value at $r = 0$ is zero, while it is strictly positive at all $r > 0$; therefore, if proposed a positive amount, he would accept. Third, the rational player 1's value at $r = 0$ is clearly maximized at that state, which means she would never propose a strictly positive amount to player 2. Fourth, player 1 will clearly accept any proposal that *gives player 2 zero: whether she accepts or rejects, the state doesn't change, so she may as well accept. Finally, suppose that player 2 offers* $(1-\epsilon, \epsilon)$*. If player 1 rejects, she gets* δ*, while if she accepts she gets* $(1-\delta)(1-\epsilon)+\frac{\delta}{2}(1-\epsilon)$*. Therefore, player 1 will reject an offer giving a vanishingly small amount to player 2 if* $\delta \geq \frac{2}{3}$*.*

Notice one feature of the above example is that the behavioural player *willingly* offers the entire pie to rational player when it is his turn to propose. Of course, the behavioural player is actually indifferent between any possible proposal he could make. Therefore, if he has even an ounce of spite or concern for fairness, one might expect him to behave differently. For example, if the behavioural player always proposes 0 at $r=0$, then the rational player's continuation value is only $\frac{\delta}{2}$ In this case, it can be shown that for every $\delta < 1$, there is an $\epsilon > 0$, such that the rational player accepts a proposal to move to the state $r = \epsilon$. Indeed, any absorbing state, $r \geq 0$, supported by the strategies of Proposition 1 for $r' > r$, can be undone, for all $\delta < 1$, merely by the behavioural player "refusing to play ball" with the rational player and proposing $(0, r)$ whenever it is his turn to propose.

While Example 1 demonstrates that there are other equilibria in which the rational player can do better than the Markov equilibrium of Proposition 1, the same cannot be said about the behavioural player. This follows because, for any $r > 0$, the behavioural player strictly prefers to accept an offer in which he obtains his reference point, r.[5] The best that the behavioural player could do is to reject with positive probability an offer of 0 when $r = 0$; however, in this case, the rational player will merely make an arbitrarily small positive offer in order for it to be accepted. In this sense, the "fair" equilibrium of Proposition 1 is actually the worst Markov equilibrium for the rational player.

3.2 Downwards Adjustable Reference Points

We now turn our attention to the case in which the reference point of the behavioural player adjusts downwards by a factor of $\alpha \in [0,1)$ following a disagreement. That is, after a disagreement in period t, $r_{t+1} = \alpha r_t$. Matters are now substantially more complicated because we must contemplate the possibility that the rational player will attempt to manipulate the reference points of behavioural, particularly if r gets too high. As before, we focus on Markov strategies.

The following result is easily seen:

Lemma 2. *In every Markov equilibrium, there exists* $\bar{r} < 1$*, such that for all* $r > \bar{r}$*, disagreement necessarily arises.*

[5]This follows because, whether the behavioural player accepts or rejects, the state will remain unchanged in the next period. Therefore, he must accept as long as a proposal of $r > 0$ is made to him.

The proof is omitted but the intuition is easily seen. If the reference point of the behavioural player is sufficiently high, then the rational player would do better to force disagreement, lower the reference point, and then propose the status quo whenever given a chance to do so. Similar to the case in which reference points are downwards fixed, it is easy to see that in any Markov equilibrium, there cannot be an interval of absorbing states.

We now show that an analog to Proposition 1 holds when references points are downwards adjustable. However, the equilibrium in this case must also account for Lemma 2. In particular, there exists $\bar{r} \in (0,1)$ such that for $r > \bar{r}$, disagreement necessarily arises. On the other hand, for $r \leq \bar{r}$, the rational player always offers the status quo, while the behavioural player proposes $g(r) \geq r$ for himself and $1 - g(r)$ for player 1. That is:

Proposition 2. *There exists a Markov equilibrium with absorbing state* $\bar{r} = \frac{2-\delta-\alpha\delta}{2(1-\alpha\delta)}$. *For* $r \leq \bar{r}$, *the rational player always proposes the status quo, while the behavioural player's proposal is given by the function* $g(r) = \frac{2-\delta(1+\alpha(1-r))}{2-\alpha\delta}$. *For* $r > \bar{r}$, *disagreement necessarily arises.*

Proof. See appendix. □

As was the case in which reference points are downwards fixed, this equilibrium has the property that it is fair, with the expected value at $r = 0$ being $\frac{1}{2}$ for both players. The main difference between the two cases is that here, the rational player does not eventually cede the entire pie to the behavioural player. Moreover, if reference points are initially high, the rational player will reject any offer that the behavioural player could make and would, herself, make an unacceptable to her opponent in order to induce disagreement. Thus the rational player, will strategically manipulate the reference point of the behavioural player. It is in this sense that we say, for example, that firms may try to *break a union* by repeatedly rejecting proposals and by also making unacceptable proposals until such a time as the reference point of the behavioural player is at a point conducive to a "more fair" agreement.

Upon examining $\bar{r}$, it is also apparent that:

Corollary 1. *The absorbing state,* $\bar{r}$ *is increasing in* α *and decreasing in* δ.

That is, as the behavioural player becomes more belligerent (*i.e.,* reference points adjust downwards more slowly following disagreement), the rational player must, eventually, cede more of the pie. Note, however, that this does not change the expected value of the rational player at $r = 0$: for any $\alpha \in (0,1)$, the rational player's expected value is $\frac{1}{2}$. This is possible because, as can also be seen, $\frac{\partial g(r)}{\partial \alpha}|_{r=0} < 0$. Therefore, the reference points increase more slowly.

Before concluding our analysis of the case in which one player has reference-dependent preferences and the other player is fully rational, observe that there may be equilibria in which the rational player obtains a disproportionately large share of the pie. As was shown in Example 1, the key to constructing such equilibria involves using the Markov equilibrium derived above as a punishment in the continuation game should reference points be allowed to increase.

4 Repeated Bargaining When Both Players Have Reference-Dependent Preferences

In this section, we turn to the case in which both players have reference dependent preferences. All other aspects of the game are unchanged. In particular, payoffs are discounted by δ each period, the utility function of both players is given by (1) and reference points adjust according to (2).

4.1 Downwards Unadjustable Reference Points

We begin with the case in which reference points do not adjust downwards following a disagreement (*i.e.,* $\alpha = 1$). Therefore, if ever an efficient offer is accepted, this effectively ends negotiations, since in the next period, reference points will sum to 1. As above, we are interested in the Markov equilibria of this game.

We first go through the intermediate step of solving for what we call each player's *unconstrained offer*, given the equilibrium strategies in the continuation game. The reason that one's unconstrained offer does not correspond to the actual offer made in equilibrium comes from the fact that reference points must be respected. Therefore, this adds another constraint that must be taken care of. The unconstrained offers can be thought of intuitively as follows. Given the continuation value induced by the equilibrium strategies, what amount would i offer to j in the current period that would make j just indifferent between accepting and rejecting, *ignoring the fact that reference points must be respected*. The constrained offer is defined as the maximum of the unconstrained offer and the reference point. We will show that the unconstrained offer functions can be characterised rather easily by the solution to a contraction mapping. We then construct the equilibrium strategy which takes into consideration the constraints given by reference points. With downwards unadjustable reference points, one can see that agreements will always be reached in every Markov equilibrium since the status quo (*i.e.,* current reference point) will always be accepted if offered. However, it may be that the reference point is actually above one's continuation value; therefore, the actual, constrained offer may be greater than the unconstrained offer.

Let $\mathcal{C}(\Delta, [0, 1])$ denote the space of continuous functions with domain $\Delta = \{x \in \mathbb{R}^2_+ :$

$x_1 + x_2 \leq 1\}$ and range $[0,1]$ and define the map $\Phi : \mathcal{C}(\Delta, [0,1])^2 \mapsto \mathcal{C}(\Delta, [0,1])^2$ by:

$$\Phi(f_1, f_2)(r) = \begin{bmatrix} \frac{\delta}{2}\left(1 + \max\{f_1(r), r_1\} - \max\{f_2(r), r_2\}\right) \\ \frac{\delta}{2}\left(1 + \max\{f_2(r), r_2\} - \max\{f_1(r), r_1\}\right) \end{bmatrix} \tag{4}$$

A fixed point, $(\bar{f}_1, \bar{f}_2)$, of this map then defines the *unconstrained* offers that the players would like to make.[6] In the appendix, we show that Φ is a contraction, but here we pause to give some intuition. Consider the term $\frac{\delta}{2}\left(1 + \max\{f_1(r), r_1\} - \max\{f_2(r), r_2\}\right)$ on the right-rand side of (4). It can be thought of as the average continuation value to player 1 if he rejects an offer, while expecting both players to play according to the *constrained* offer functions in the next period. Following the rejection, reference points do not adjust. Then with probability 0.5, player 2 will be selected to propose and will propose $\max\{f_1(r), r_1\}$ to player 1. On the hand, with probability 0.5, player 1 will be chosen to proposes and will offer $\max\{f_2(r), r_2\}$, keeping the remainder for himself. Of course, this average payoff is then discounted by δ.

In the appendix, we show that the unique solution to (4) gives us a Markov equilibrium. That is:

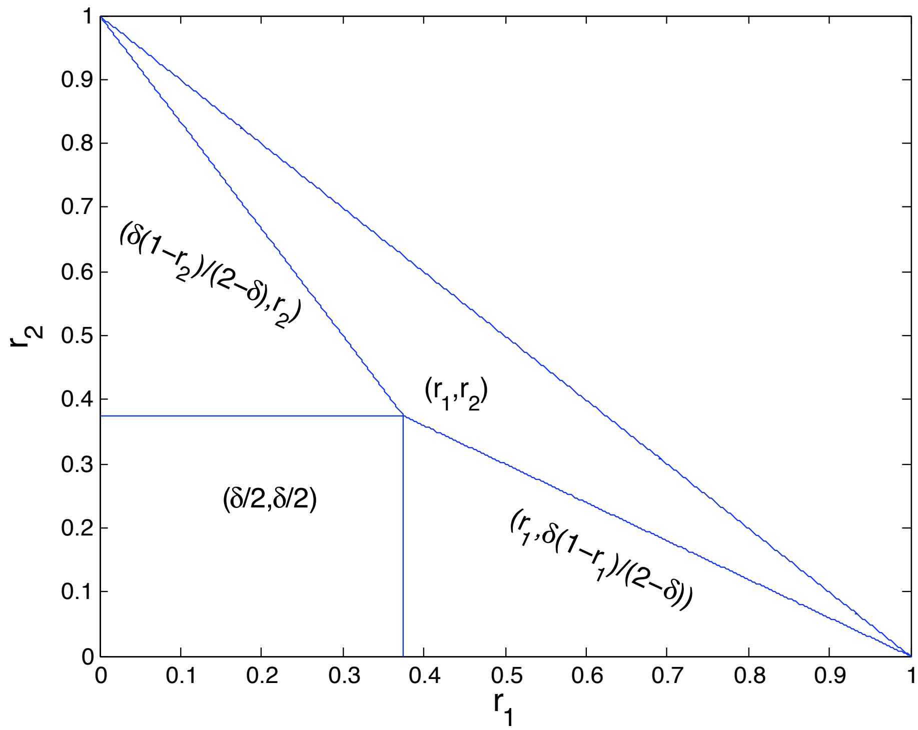
Proposition 3. *There is a pure strategy Markov equilibrium of the bargaining game with downwards unadjustable reference points. In this equilibrium, player i proposes $\max\{f_j(r), r_j\}$ and accepts offers greater than or equal to $\max\{f_i(r), r_i\}$, where (f_1, f_2) is the unique solution to $\Phi(f_1, f_2) = (f_1, f_2)$.*

Proof. See appendix. □

Indeed, if one solves (4), it is possible to see that the bargaining space gets divided into four regions, depending on where reference points lie. In Figure 1 we show the offers that each of the players make to the other in each of the four different regions. The first element of each tuple is the offer made by player 2 to player 1 if reference points fall within the given region, while the second element is the offer made by player 1 to player 2 if reference points fall within the given region. As can be seen, when the reference points of both players are sufficiently low (*i.e.,* less than $\frac{\delta}{2}$), then each player proposes $\frac{\delta}{2}$ to his/her opponent and takes the remainder of the pie for him/herself. On the other hand, when one player's reference point is above $\frac{\delta}{2}$, then, in equilibrium, this player, must be given her at least her reference point. Because of this, the other player's continuation value decreases, which leads her to receive lower offers — at least when her reference point is not too high. Finally, when both players' reference point are high enough, then each players reference point is above her continuation value, meaning that they must be offered their reference point to accept.

[6] In particular, $\bar{f}_i$ denotes the amount (ignoring reference points) that j would like to give to i.

Figure 1: Equilibrium Offers Given Reference Points ($\delta = \frac{3}{4}$)



Note: The first (resp. second) element in each tuple represents the offer made by player 2 (resp. player 1) to player 1 (resp. player 2) if the reference points fall within the particular region.

Three further brief remarks are in order. First, for $(r_1, r_2) \in [0, \frac{\delta}{2}]$, $f_i(r) = \frac{\delta}{2}$, which is identical to that of the standard random proposer Rubinstein bargaining model, while outside of this region, the reference point of at least one player, say player i, will bind. Second, notice also that in this model, reference points only affect the outcome, they do not lead to disagreements. In particular, players do not engage in the strategic manipulation of their opponent's reference point. Finally, observe that the equilibrium is efficient since an all equilibrium offers exhaust the entire pie and are accepted.

4.2 Downwards Adjustable Reference Points

We now discuss the case of downwards adjustable reference points (*i.e.,* $\alpha < 1$ in (2)). Our immediate goal is to prove the existence of a Markov equilibrium similar to that of the previous subsection. Our method for constructing such an equilibrium follows the same procedure — in particular, we again define the unconstrained offer function as the unique fixed point to a particular contraction mapping. However, in this case, more care must be taken. As happened in the case of downwards adjustable reference points with one rational player and one behavioural player, here too, if reference points are too high, players might attempt to manipulate the reference point of their opponent in order to secure a more favourable division of the pie in a future period.

Let $\mathcal{C}(\Delta, [0,1])$ denote the space of continuous functions with domain Δ and range $[0,1]$ and define the map $\Psi : \mathcal{C}(\Delta, [0,1])^2 \mapsto \mathcal{C}(\Delta, [0,1])^2$ by:

$$\Psi(f_1, f_2)(r) = \begin{bmatrix} \frac{\delta}{2}\left(1 + \max\{f_1(\alpha r), \alpha r_1\} - \max\{f_2(\alpha r), \alpha r_2\}\right) \\ \frac{\delta}{2}\left(1 + \max\{f_2(\alpha r), \alpha r_2\} - \max\{f_1(\alpha r), \alpha r_1\}\right) \end{bmatrix} \tag{5}$$

A fixed point, $(\bar{f}_1, \bar{f}_2)$, of this map then defines the *unconstrained* offers that the players would like to make. In much the same way when reference points were downwards fixed, one can easily show that Ψ is a contraction mapping. Therefore, the Banach Fixed Point Theorem guarantees the existence of a unique fixed point.

However, one important difference between the present case, where $\alpha < 1$, and the previous case, where $\alpha = 1$, is that the unconstrained offer functions need not be valid over all of Δ. In particular, suppose that $f_i(r) + r_j > 1$ and that player j is selected to propose. In order for player i to accept, she must be offered at least her continuation value of $f_i(r)$, while in order for j to be happy with any agreement, he must receive at least r_j. However, both of these constraints cannot be satisfied when $f_i(r) + r_j > 1$, which means that disagreement must necessarily arise. Therefore, we can divide the bargaining space into two regions: an agreement region and a disagreement region. Specifically,

Definition 1. *Define the region of agreements as the set* $\Delta^a(\alpha, \delta) = \{r \in \Delta : r_1 \leq 1 - f_2(r) \;\&\; r_2 \leq 1 - f_1(r)\}$.

Definition 2. *Define the region of disagreements as the set* $\Delta^d(\alpha, \delta) = \Delta \setminus \Delta^a(\alpha, \delta)$.

Within the set $\Delta^a(\alpha, \delta)$, immediate agreement will arise with player i proposing $\max\{f_j(r), r_j\}$ to player j, and accepting any offer greater than or equal to $\max\{f_i(r), r_i\}$. As we will presently show, the set of *absorbing states* is simply the Pareto frontier of $\Delta^a(\alpha, \delta)$. Define this set as $\mathcal{A}(\alpha, \delta)$. In the region $\Delta^d(\alpha, \delta)$, disagreement necessarily arises.

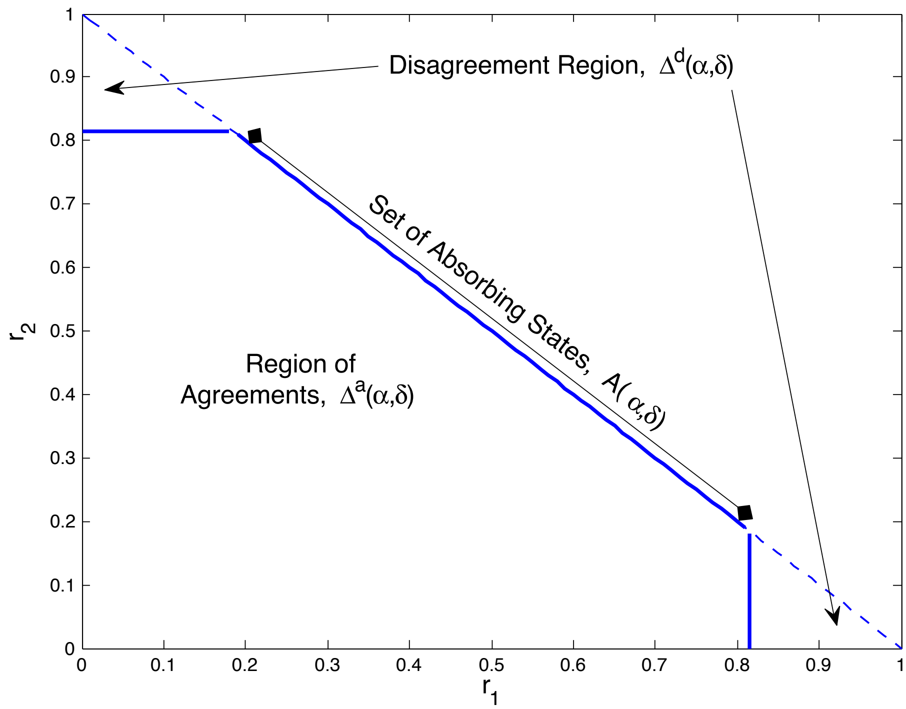
In the appendix, we prove the following result:

Proposition 4. *There exists a symmetric, Markov equilibrium to the bargaining game with downwards adjustable reference points — the strategies of which can be derived by the solution to (5). In this equilibrium, disagreement arises whenever* $r \in \Delta(D)$.

Proof. See appendix. ☐

In Figure 2, we partition the bargaining space into the agreement and disagreement region for a particular example. As can be seen, if reference points are less than, approximately, 0.8154, then agreements are possible. On the other hand, if one of the player's reference

FIGURE 2: The Bargaining Region: $\delta = \frac{3}{4}$ & $\alpha = \frac{9}{10}$



point is above this threshold, then one or more periods of disagreement must take place in order for reference points to adjust downwards into $\Delta^a(\alpha, \delta)$.

Although Figure 2 shows when agreements are possible and when they are not, it does not indicate what the offers actually are when agreements are possible. It is easy to see that if $(r_1, r_2) \in [0, \frac{\delta}{2}]^2$, then each player will offer their opponent $\frac{\delta}{2}$ and take the remainder for him/herself. Outside of this region, the reference point of at least one player must bind, and if both reference points are sufficiently high, then the reference points of both players become the binding offers. Finally, notice that, given the parameters of the model for the example depicted in Figure 2, if $r_i \approx 1$, disagreement will last two periods. For a general α and δ, for extreme initial reference points, disagreement may well last for a multiple, but finite, number of periods. It is also interesting to note that, for some values of α and δ, a player may actually prefer to reject an extremely generous offer that is off the equilibrium path.[7] This is done so as to avoid the prospect of a protracted period of disagreement in the future.

Turn now to the comparative statics of $\Delta^a(\alpha, \delta)$. In particular, how does $\Delta^a(\alpha, \delta)$ change if either α (the speed at which reference points adjust downwards following disagreement) or δ (the discount factor) changes. Intuition suggests that if α increases, then $\Delta(A)$ should

[7]Our numerical calculations indicate that this is likely to arise when δ is high relative to α, and α is not too small (in particular, $\alpha > \frac{\delta}{2}$).

also increase. This follows because with a higher α, reference points adjust downwards by a smaller amount following a disagreement. Therefore, a player who would like to improve her relative standing by forcing a disagreement has less to gain by doing so. On the other hand, we would expect $\Delta(A)$ to shrink as δ increases. For a higher δ, the future now weighs more heavily, which means that forcing a disagreement in a single period and, therefore, obtaining a larger share of the pie in all future periods is now more attractive. Indeed, this intuition holds, as we demonstrate with the following results:

Proposition 5. *If $\alpha_1 < \alpha_2$ then $\Delta^a(\alpha_1, \delta) \subseteq \Delta^a(\alpha_2, \delta)$.*

Proof. See appendix. □

Proposition 6. *If $\delta_1 > \delta_2$ then $\Delta^a(\alpha, \delta_1) \subseteq \Delta^a(\alpha, \delta_2)$.*

Proof. See appendix. □

Example 2. *It is easy to see that when $\alpha < \frac{\delta}{2}$, $\Delta^a(\alpha, \delta) = \{r \in \Delta : r_1 + r_2 \leq 1, r_i \leq 1 - \frac{\delta}{2}\}$ and the set of absorbing states is given by $\mathcal{A}(\alpha, \delta) = \{x \in \Delta : x_1 + x_2 = 1, x_i \in [\frac{\delta}{2}, 1 - \frac{\delta}{2}]\}$. Moreover, the equilibrium strategies are very simple: For all $r \in \Delta^a(\alpha, \delta)$, $f_i(r) = \frac{\delta}{2}$. That is, for $r \in \Delta^a(\alpha, \delta)$, each player offers* $\max\{r_j, \frac{\delta}{2}\}$ *and keeps the remainder for him/herself. We will use this below to demonstrate that there is not a unique Markov equilibrium.*

4.3 Lack of Uniqueness: Supporting Isolated Points

The Markov equilibrium embodied in Proposition 4, intuitive and has a number of reasonable properties. However, the result makes no claim as to the uniqueness of the equilibrium. Indeed, it is not difficult to show that there exist other Markov equilibria of the infinitely repeated bargaining game. We now describe one such equilibrium.

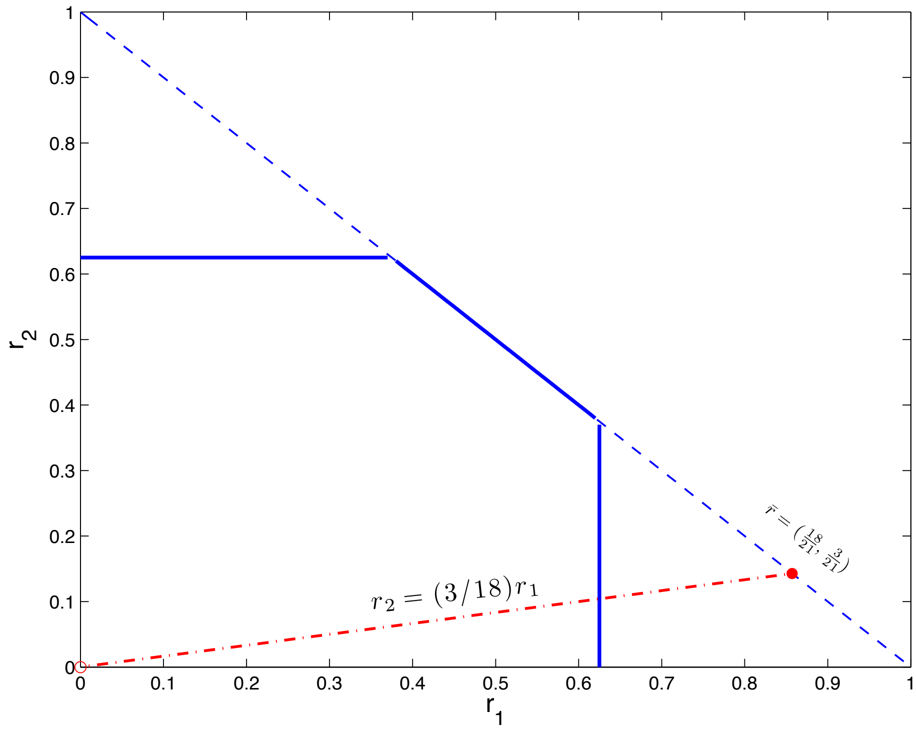
Suppose that $0 < \alpha \leq \frac{3}{8} < \delta = \frac{3}{4}$ so that, for the equilibrium described by Proposition 4, the set of absorbing states is given by $\mathcal{A}(\alpha, \delta) = \{(r_1, r_2) : r_1 + r_2 = 1, r_i \in [\frac{3}{8}, \frac{5}{8}]\}$. We will now show that, with a relatively simple modification, we can support $\bar{r} = (\frac{18}{21}, \frac{3}{21})$ as an absorbing state.

In Figure 3, the agreement region, given that subjects play the Markov equilibrium of Proposition 4 is the region enclosed by the solid lines. Let σ denote the Markov equilibrium strategy profile. In order to support $\bar{r}$, suppose that both players follow σ everywhere on Δ, *except* on the set $\Delta'(\bar{r}) = \{(r_1, r_2) \in \Delta : r_2 = \frac{3}{18} r_1 \,\&\, r_1 > 0\}$. This is the $-.$ line in Figure 3. We modify the strategies in the following way:

1. Given an initial state $r \in \Delta'(\bar{r})$, each player proposes $\bar{r}$.

2. From any initial state $r \in \Delta$, player 1 accepts all offers $x \in \Delta'(\bar{r})$ such that $x_1 \geq r_1$ and will reject any other offer.

FIGURE 3: Supporting Isolated Points in Equilibrium



3. From any initial state $r \in \Delta'(\bar{r})$, player 2 accepts any offer $x_2 \geq r_2$, while from any initial state $r \notin \Delta'(\bar{r})$, player 2 rejects any proposal $x \in \Delta'(\bar{r})$.

In all other instances, players follow the strategy given by σ, the Markovian equilibrium defined by Proposition 4.

First, observe that, given the acceptance strategies of the players, the offers are clearly optimal. Second, it is clearly optimal for player 1 to accept a proposal of $\bar{r}$. With a little effort, we can also see that player 1 rejects any proposal $x \notin \Delta'(\bar{r})$. To see this, suppose that player 1 was offered. The expected value of accepting this proposal is $(1-\delta)+\delta/2 = 5/8$. On the other hand, by rejecting the proposal, player 1's expected value is $\delta(18/21) = 9/14 > 5/8$. Thus, player 1 prefers to reject.

Now consider player 2. It is clearly optimal for player 2 to accept any offer $x \in \Delta'(\bar{r})$ such that $x_2 \geq r_2$. On the other hand, suppose that player 2 is offered $x \notin \Delta'(\bar{r})$ such that $x_2 \geq r_2$. By accepting, the worst that player 2 can do is $(1-\delta)x_2 + \delta^2/2 = x_2/4 + 9/32$, while the value from rejecting is $\delta(3/21) = 3/28 < x_2/4 + 9/32$ *for any* $x_2 \geq 0$. Thus, it is optimal for player 2 to accept.

Finally, we must specify the behaviour of players if, at a given state $r \notin \Delta'(\bar{r})$ receive an offer $x \in \Delta'(\bar{r})$ such that $x_i \geq r_i$. It is easy to see that player 2 would reject such offers, while player 1 prefers to accept. Thus, we have shown that $\bar{r}$ can be supported as an absorbing state in Markov equilirbium.

Notice that equilibria of this type suffer from a number of undesirable properties, which, we believe, make it not very robust. First, the region $\Delta'(\bar{r})$ is very knife-edge. In particular, if the evolution of reference points were subject to small, random trembles, the *weak* player would strictly prefer to reject any offer in $\Delta'(\bar{r})$ because upon doing so, with probability 1, reference points would leave the region $\Delta'(\bar{r})$ and from this point on, she could expect to get half of the pie. Second, if $\alpha = 0$ or if there were some other way for reference points to eventually reach 0 following a series of rejections, the equilibrium would also no longer exist.[8] Third, the strategies which support this asymmetric equilibrium are discontinuous in the state space.

Ideally, we would like to show that the Markov equilibrium characterised by Proposition 4, is unique in some more restricted class. For example, one conjecture is that the equilibrium is unique in the class of strategies which are continuous functions of the state. Unfortunately, we have been unable to prove this result. Introducing some randomness into the evolution of reference points could also be expected to eliminate the type of equilibria that we have shown to exist. One can easily show that there is a unique equilibrium in the finite horizon game, and as the time horizon grows arbitrarily long, the equilibrium converges to the Markov equilibrium described in Proposition 4.

Define the unconstrained offer function in period n recursively as follows

$$f_i^n(r) = \frac{\delta \sum_{k=0}^{T-n-1} \delta^k}{2 \sum_{k=0}^{T-n} \delta^k} \left[\max\{f_i^{n+1}(\alpha r), \alpha r_i\} + 1 - \max\{f_j^{n+1}(\alpha r), \alpha r_j\} \right] \tag{6}$$

with $f_i^T(r) = 0$ for all r is the unconstrained offer in the final period. In the finite horizon game, the agreement region in period n is then $\Delta(A, n) = \{r \in \Delta \,:\, r_j \leq 1 - f_i^n(r),\, i \neq j\}$. Observe that as $T \to \infty$, (6) converges to $f_i(r)$ — the equilibrium unconstrained offer function of the infinite horizon game.

5 Related Literature

Compte and Jehiel [3] consider a model of bargaining with reference points as we do; however, there are differences between our two models. First, Compte and Jehiel [3] consider the problem of dividing a single pie, while we consider a repeated environment in which players must divide a pie in each of an infinite number of periods. A second, and more fundamental, difference concerns the issue of how exactly reference points are updated. In our model,

[8] Note that modifying the evolution of reference points so that they may reach zero (from above) is not a panacea; one can show the existence of other asymmetric, unequal equilibria. However, these equilibria will also suffer from the other two criticisms raised here.

reference points are updated as a function of past consumption. Moreover, reference points are adjusted in every period t. Compte and Jehiel [3] consider a bargaining protocol in which, at each period, there is an exogenous risk of breakdown. If a breakdown occurs, reference points are adjusted and a new bargaining *round* begins upon paying a cost c. Before each new *round* begins, reference points adjust according to a weighted function of the current reference point and the **best** offer received over the entire course of bargaining. *A priori* it is not clear how reference points should adjust — based on past realisations of consumption or on past offers.

Li [7] presents a model in which reference points adjust after every offer. In his basic set-up, a player strictly prefers rejection to any offer which gives a discounted utility lower than the discounted utility of the most generous previously rejected offer. For example, if the most generous offer (in discounted utility terms) up to time t made to player 1 is $\delta^{t-1}\alpha$, player 1 will reject any offer $\alpha' < \frac{\alpha}{\delta}$. Therefore, in the (essentially) unique SPE of Li's model, offers start small and gradually increase until a *clinching* offer is made.

If we allow reference points to adjust after every offer in Compte and Jehiel [3], we might expect results similar to Li [7]. In particular, bargaining must still be a gradual process — initial offers cannot be too generous because the strategic role of rejection is still at play. Thus bargaining must be a gradual process. This is in contrast to our model of downwards fixed reference points when both players have reference-dependent preferences in which agreements occur immediately. This follows due to our assumption that reference points adjust only after actual consumption.

6 Conclusion

In this paper we have studied the problem of bargaining by otherwise rational agents, but for the presence of reference points. We considered both the case in which a fully rational player was matched with a player with reference-dependent preference and the case in which two players, both with reference-dependent preferences, were matched. For each case, we also allowed reference points to be either fixed or to adjust downwards following a disagreement.

Regardless of how reference points adjust following disagreement, when a rational player repeatedly bargains with a behavioural player, we showed that there is a Markov equilibrium in which the rational player gradually cedes her share of the pie to the behavioural player. When reference points do not adjust following disagreement, this process does not end until, in the limit, the rational player has ceded the entire pie. In contrast, when reference points do adjust downwards following disagreement, then the strategic role of disagreement comes into play and moderates what the behavioural player can hope to extract from the rational player.

In the text, we argued that the Markov equilibria characterised in the text are actually the *worst* Markov equilibria for the rational player, despite the fact that they are fair (when summed over the players' infinite-length interaction). We also showed by means of an example, that there may be *commitment* equilibria in which the rational player obtains a strictly higher payoff. The key to the construction of these equilibria is to use the rational player's worst Markov equilibrium as a punishment if ever the reference point of the behavioural player is allowed to increase beyond a certain point.

One wonders whether there is more than a hint of realism to the equilibria described in this part of the text. Indeed, it seems that the history of labour relations between firms and workers generally begins in favour of the firm, while labour, over-time, gradually wrests concessions from the firm to improve their situation. Once conceded, this is the new status quo for all future bargaining, and it is difficult for the firm regain ground. Indeed, for much of the history between the auto workers union and the so-called Big 3 automakers, this seems like a particularly apt description. It is only recently that the Big 3 have been able to gain concessions from the auto workers and only with the threat of bankruptcy and liquidation due to a sharp downward (and persistent) drop in demand. On the other hand, Wal-Mart, with its fierce opposition to unionisation, seems to be playing one of the commitment equilibria that we described.

We also studied the situation in which both players had reference-dependent preferences. Rather than the gradual bargaining process in which the rational player ceded her share of the pie, in this case, once an agreement is made, it exhausts the entire pie and will remain in force for the entire duration of their interaction. In particular, when reference points do not adjust downwards following a disagreement, we have shown that their presence does not lead to disagreements but instead leads to potentially different bargaining outcomes relative to the case in which reference points do not exist. Therefore, reference points are akin to outside options, though in our model, they are determined endogenously through the bargaining process itself.

In contrast, when reference points do adjust downwards following a disagreement, players may prefer to force disagreement in order to reduce inequality between the two players. As part of our construction, we characterised the bargaining region for a specific Markov equilibrium and showed that it had some intuitive properties: when reference points are not too unequal, an efficient agreement will always be reached. On the other hand, when reference points are sufficiently unequal, disagreement must necessarily arise for a number of periods until they have entered the agreement region. Moreover, the size of the agreement region has intuitively appealing properties. For a fixed speed of adjustment, as players become more patient, the agreement region shrinks, while for a fixed level of patience, as reference points adjust downwards more quickly, the agreement region also shrinks.

There are a number of extensions worth exploring. All of our analysis was concerned with a constant pie; however, it would be interesting to study the stochastic pie case. While interesting, such an analysis would also be quite difficult since it introduces another state variable. We do know that Markov equilibria of the two-person bargaining model is less efficient than a single-decision maker. In particular, in the two-person model, players always consumer a larger portion of the maximum possible pie than would a lone decision maker. Thus disagreements, due to infeasible pies, would be expected to be more frequent. Beyond that, we do not know what would happen. We have also assumed that players have complete information about the reference point of their opponent and the adjustment process reference points. Introducing some uncertainty on either of these fronts is also worth exploring, though perhaps in a finitely repetition setting.

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A Proofs

A.1 Proof of Proposition 1

Since players employ Markov strategies, it is clear that player 2 will accept a status quo proposal of r_t. Also, given the proposed strategies, it is clear that the equilibrium value to player 1, the rational player, is decreasing in the reference point of player 2. Therefore, the best that player 1 can do is propose $(r_t, 1-r_t)$.

Next, let $V(r,i)$ denote the value to player 1 when the reference point is r and player $i \in \{1,2\}$ is selected to propose. Under the proposed strategies, it can easily be seen that:

$$V(r,1) = \frac{2(1-\delta)(1-r)}{2-\delta} + \frac{\delta}{2-\delta} V(r,2).$$

Next, observe that player 2 will make a proposal $1-y$ such that $y \geq r$ and:

$$\begin{aligned} V(r,2) &= (1-\delta)(1-y) + \frac{\delta}{2}\left[V(y,1) + V(y,2)\right] \qquad (7) \\ &= \frac{\delta}{2}\left[V(r,1) + V(r,2)\right]. \qquad (8) \end{aligned}$$

That is, player 2 pushes player 1 down to her continuation value from rejecting the proposal. From (8), we are able to determine that $V(r,2) = \frac{\delta}{2}(1-r)$. Using (7), we are then able to solve for y. Indeed, y is the solution to:

$$\frac{\delta}{2}(1-r) = (1-\delta)(1-y) + \frac{\delta}{2}(1-y),$$

where the right-hand side of the above equation is given by (7), upon noting that $V(y,2) = \frac{\delta}{2}(1-y)$. Solving this equation gives us $y = r + \frac{2(1-r)(1-\delta)}{2-\delta} \equiv f(r)$.

Finally, observe that $f(r) \geq r$, $f'(r) \in (0,1)$ and $f(1) = 1$. Therefore, as $t \to \infty$, $r_t \to 1$. □

A.2 Proof of Lemma 1

Suppose to the contrary that there exists an equilibrium such that there exists an interval $\mathcal{R} = (\underline{r}, \bar{r})$ such that all $r \in \mathcal{R}$ are absorbing. Next, suppose that $r, r' \in \mathcal{R}$ with $r' > r$. Observe that, from the state r, player 1 will accept a proposal to move to state r' provided that:

$$1 - r' \geq \delta(1 - r),$$

which can be rearranged to $r' - r \leq (1-\delta)(1-r)$. Clearly, therefore, this inequality must be satisfied for some $r' \in \mathcal{R}$, contradicting the assumption that $r \in \mathcal{R}$. □

A.3 Proof of Proposition 2

As before, let $V(r, i)$ denote the value to player 1 when she player $i \in \{1, 2\}$ is selected to propose at state r. Then, for $r \leq \bar{r}$, we have:

$$\begin{aligned} V(r,2) &= (1-\delta)(1-g(r)) + \frac{\delta(1-\delta)(1-g(r))}{2-\delta} + \frac{\delta}{2-\delta}V(g(r),2) \qquad (9) \\ &= \max\{1-\bar{r}, \frac{\delta(1-\delta)(1-\alpha r)}{2-\delta} + \frac{\delta}{2-\delta}V(\alpha r, 2)\}, \qquad (10) \end{aligned}$$

while for $r > \bar{r}$,

$$V(r,1) = V(r,2) = \frac{\delta}{2}(V(\alpha r, 1) + V(\alpha r, 2)).$$

Observe that (10) differs from its counterpart in the downwards fixed case by the presence of the max expression. When references points are downwards fixed, it is always optimal for player 2 to increase his own reference point as high as possible given player 1's continuation value. However, when reference points adjust downwards following a disagreement, player 2's behaviour is kept somewhat in check. Therefore, rather than making player 1 indifferent between accepting and rejecting, in some cases, player 1 may find it optimal to simply propose a move to the absorbing state. Finally, note that $\bar{r}$ is obtained by using (9) in order to solve for $g(r)$, and in particular, $\bar{r}$ is a fixed point of $g(r)$.

We now show that the above system is solved with a decreasing, linear function $V(r, 2)$. First, observe that $V(0, 2) = \frac{\delta}{2}$. Second, using (10), ignoring the max expression, and under the assumption of linearity, one can show that $V(r, 2) = \frac{\delta}{2} - \frac{\alpha\delta(1-\delta)}{2-\delta-\alpha\delta}r$. Third, using (9), and our expression for $V(r, 2)$, we can solve for $g(r)$ and $\bar{r}$. In particular, it can be shown that $g(r) = \frac{2-\delta(1+\alpha(1-r))}{2-\alpha\delta}$ and $\bar{r} = \frac{2-\delta-\alpha\delta}{2(1-\alpha\delta)}$.

In order to show that this is an equilibrium, we must show that $\frac{\delta(1-\delta)(1-\alpha r)}{2-\delta} + \frac{\delta}{2-\delta}V(\alpha r, 2) \geq 1 - \bar{r}$, since we implicitly made this assumption in calculating $V(r, 2)$. We must also show that when $r > \bar{r}$, player 1 actually prefers to reject any proposal. Regarding the former, after tedious algebra, it can be show that the inequality holds strictly for $r < \bar{r}$ and with equality at $r = \bar{r}$. Regarding the latter point, again, after some tedious algebra, it can be shown that for $r > \bar{r}$, the rational player 1 strictly prefers to reject a proposal giving her $1 - r$, while for $r = \bar{r}$, player 1 is indifferent between accepting and rejecting. Note also that while player 1 might accept a proposal which gives her more than $1 - r$, given our behavioural assumption about player 2, such a proposal would never be made. □

A.4 Proof of Proposition 3

As in the main text, define the mapping: $\Phi : \mathcal{C}(\Delta, [0,1])^2 \mapsto \mathcal{C}(\Delta, [0,1])^2$ by:

$$\Phi(f_1, f_2)(r) = \begin{bmatrix} \frac{\delta}{2}\left(1 + \max\{f_1(r), r_1\} - \max\{f_2(r), r_2\}\right) \\ \frac{\delta}{2}\left(1 + \max\{f_2(r), r_2\} - \max\{f_1(r), r_1\}\right) \end{bmatrix}$$

We first show that this mapping is actually a contraction. Let $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathcal{C}(\Delta, [0,1])^2$. Define $d(f,g) = \sup_{r\in\Delta} |f_1(r) - g_1(r)| + \sup_{r\in\Delta} |f_2(r) - g_2(r)|$. We now calculate $d(\Phi(f), \Phi(g))$; indeed, we have:

$$\begin{aligned} d(\Phi(f), \Phi(g)) &= \delta \sup_{r\in\Delta} |\max\{f_1(r), r_1\} - \max\{g_1(r), r_1\} + \max\{f_2(r), r_2\} - \max\{g_2(r), r_2\}| \\ &\leq \delta \sup_{r\in\Delta} |\max\{f_1(r), r_1\} - \max\{g_1(r), r_1\}| \\ &+ \delta \sup_{r\in\Delta} |\max\{f_2(r), r_2\} - \max\{g_2(r), r_2\}| \end{aligned}$$

It is easily seen that $\delta \sup_{r\in\Delta} |\max\{f_1(r), r_1\} - \max\{g_1(r), r_1\}| \leq \delta \sup_{r\in\Delta} |f_1(r) - g_1(r)|$ and similarly, $\delta \sup_{r\in\Delta} |\max\{f_2(r), r_2\} - \max\{g_2(r), r_2\}| \leq \delta \sup_{r\in\Delta} |f_2(r) - g_2(r)|$. Therefore, $d(\Phi(f), \Phi(g)) \leq \delta d(f,g)$, which proves that Φ is a contraction mapping and so has a unique fixed point.

Now, let $f_1(r)$ and $f_2(r)$ denote the solution to the above contraction. We show that it is optimal for player i to propose $\max\{f_j(r), r_j\}$ and to accept any proposal weakly greater than $\max\{f_i(r), r_i\}$. First, consider the case in which $f_j(r) > r_j$. In this case, if player i offers $f_j(r)$ to player j, j will clearly accept since she is receiving her continuation value, which is larger than her current reference point. Next, if $r_j > f_j(r)$, it is again clear that j will accept a proposal of r_j or higher. In this case, j is getting strictly more than her continuation value, but in order for her to accept, she must be given an amount equal, at least, to her reference point.

Finally, we must show that player i, in fact, finds it optimal to propose $\max\{f_j(r), r_j\}$ to player j, and consequently taking $1 - \max\{f_j(r), r_j\}$ for himself. This is trivially true if

$f_j(r) \geq r_j$. Therefore, suppose that $r_j > f_j(r)$. In this case, we must show that $1 - r_j \geq \frac{\delta}{2}(1 + \max\{f_i(r), r_i\} - r_j)$. There are two cases to check. First, if $r_i \geq f_i(r)$. In this case, we have:

$$1 - r_j \geq \frac{\delta}{2}(1 + r_i - r_j)$$

since $r_i \leq 1 - r_j$ and $\delta < 1$. Next, we must check that the inequality holds when $r_i < f_i(r)$. In this case we have:

$$1 - r_j \geq \frac{\delta}{2}(1 + f_i(r) - r_j),$$

which holds because in this case, $f_i(r) = \frac{\delta}{2-\delta}(1 - r_j) < 1 - r_j$. Finally, any other acceptable proposal is no optimal for player i because then player i's expected future share of the pie can only be *less* and it further extends the time to which the entire pie is exhausted, essentially ending meaningful negotiations. □

A.5 Proof of Proposition 4

The proof that the mapping defined by (5) is a contraction is similar to that of Proposition 3 and is, therefore, omitted.

Let (f_1, f_2) denote the fixed point of this mapping and let $\Delta^a(\alpha, \delta)$ be defined as in the text. In order for the equilibrium functions, (f_1, f_2) to be valid in $\Delta^a(\alpha, \delta)$, it must be that $1 - \max\{f_j(r), r_j\} \geq \max\{f_i(r), r_i\}$; that is, there must be enough of the pie left over to accommodate player i's *demand*. To show this, we consider three cases. First, suppose that $r_k \geq f_k(r)$ for $k = 1, 2$. We must then show that $1 \geq r_1 + r_2$, but this is trivially true. Second, consider the case in which $f_k(r) \geq r_k$ for $k = 1, 2$. Here we must show that $1 \geq f_1(r) + f_2(r)$. From (5), we see that $f_1(r) + f_2(r) = \delta < 1$. Finally, consider the case in which $f_1(r) > r_1$ but $r_2 > f_2(r)$. Our goal here is to show that $1 \geq f_1(r) + r_2$. Subtracting r_2 from both sides gives us $1 - r_2 \geq f_1(r)$, which is one of the defining characteristics of $\Delta^a(\alpha, \delta)$.[9]

The rest of the proof follows an identical procedure as the proof of Proposition 3 and is also omitted. □

A.6 Proof of Proposition 5

The proof proceeds in a number of steps. We first need to characterise the bargaining region and partition it into the agreement region and the disagreement region. As a first step, we have the following:

Lemma 3. *For $r \in \Delta$ such that $r_i \leq \frac{\delta}{2}$, player i offers $(1 - \frac{\delta}{2}, \frac{\delta}{2})$, while player j accepts $\frac{\delta}{2}$ and above.*[10]

[9]Of course, a symmetric argument holds for the case of $r_1 > f_1(r)$ and $f_2(r) > r_2$.

[10]There may be some upper bound for which j may actually reject. For example, if j was offered approx-

Proof. Clearly, $f_i(r) = \frac{\delta}{2}$ for $i = 1, 2$ is a fixed point of (5) for $r \in [0, \frac{\delta}{2}]^2$. Furthermore, $1 - r_j - \frac{\delta}{2} > 0$ for all $r \in [0, \frac{\delta}{2}]^2$; therefore, the offer $(\frac{\delta}{2}, 1 - \frac{\delta}{2}) \in \Delta(A)$. That the offer is optimal is also clearly seen. □

Notice, therefore, that $\frac{\partial f_k(r)}{\partial r_1} = 0$ for $k = 1, 2$ and for all $r \in [0, \frac{\delta}{2}]^2$; moreover, $(\frac{\delta}{2}, 1 - \frac{\delta}{2}) \in \Delta(A)$. Next define the point $\hat{r}$ such that $\hat{r}_1 = 1 - \hat{r}_2 = f_1(\hat{r})$ and the corresponding region $\bar{\mathcal{R}} = [0, \hat{r}_1] \times [0, \hat{r}_2]$. Since $\frac{\partial f_k(r)}{\partial r_1} = 0$ for $k = 1, 2$ and for all $r \in [0, \frac{\delta}{2}]^2$, we know that $\hat{r}_1 \leq \frac{\delta}{2}$. We now claim the following:

Lemma 4. *For all $r \in \bar{\mathcal{R}}$, $\frac{\partial f_1(r)}{\partial r_1} = 0$.*

Proof. This follows easily. First, in the region $\bar{\mathcal{R}}_1 = [0, \hat{r}_1] \times [0, \frac{\delta}{2}]$, we know that $\frac{\partial f_k(r)}{\partial r_1} = 0$ for $k = 1, 2$. Now consider a point $r \notin \bar{\mathcal{R}}_1$ such that $\alpha r \in \bar{\mathcal{R}}_1$. In this case, $f_1(r) = \frac{\delta}{2}\left[1 + f_1(\alpha r) - f_2(\alpha r)\right]$. Therefore, it is easily seen that $\frac{\partial f_1(r)}{\partial r_1} = 0$. By induction, the result holds for all $r \in \bar{\mathcal{R}}$. □

We are now ready to provide a comparative static analysis of an infinitesimal change in α on $\Delta(A)$. To do this, we see how r_1 changes at the point such that $\hat{r}_1 = 1 - \hat{r}_2 = f_1(\hat{r})$. By the Implicit Function Theorem, we have that:

$$\frac{\partial r_1}{\partial \alpha}\Big|_{r=\hat{r}} = -\frac{\frac{\partial f_1(r)}{\partial \alpha}}{1 - \frac{\partial f_1(r)}{\partial r_1}} = \frac{\partial f_1(r)}{\partial \alpha} \tag{11}$$

where the second equality follows from Lemma 4. We have the following:

Lemma 5. *For $r_i \lesseqgtr r_j$, $\frac{\partial f_i(r)}{\partial \alpha} \lesseqgtr 0$.*

Proof. By Lemma 3, we have that for all $r \in \bar{\mathcal{S}}_1 = [0, \frac{\delta}{2}]^2$, $\frac{\partial f_i(r)}{\partial \alpha} = 0$. Next define the region $\bar{\mathcal{S}}_2 = \{r \in \Delta \,|\, r \notin \bar{\mathcal{S}}_1, \, \alpha r \in \bar{\mathcal{S}}_1\}$ and consider a point $r \in \bar{\mathcal{S}}_2$. In this case, $f_i(r) = \frac{\delta}{2}\left[1 + f_i(\alpha r) - f_j(\alpha r)\right]$. Therefore, it is easy to see that $\frac{\partial f_i(r)}{\partial \alpha} = 0$ for all $r \in \bar{\mathcal{S}}_2$.

In general, define the region $\bar{\mathcal{S}}_n = \{r \in \Delta \,|\, r \notin \bar{\mathcal{S}}_{n-1}, \, \alpha r \in \bar{\mathcal{S}}_{n-1}\}$. Here we have that $f_i(r) = \frac{\delta}{2}\left[1 + \max\{f_i(\alpha r), \alpha r_i\} - \max\{f_j(\alpha r), \alpha r_j\}\right]$. Note the following, if $r_i < r_j$ and $\alpha r_i \geq f_i(\alpha r)$, then it must also be that $\alpha r_j \geq f_j(\alpha r)$. Given this, by induction, the claim follows. □

Finally, since at the point $\hat{r}$ such that $\hat{r}_1 = 1 - \hat{r}_2 = f_1(\hat{r})$, $\hat{r}_1 < \hat{r}_2$, we have that $\frac{\partial f_1(r)}{\partial \alpha} \leq 0$. Therefore,

$$\frac{\partial r_1}{\partial \alpha}\Big|_{r=\hat{r}} = -\frac{\frac{\partial f_1(r)}{\partial \alpha}}{1 - \frac{\partial f_1(r)}{\partial r_1}} = \frac{\partial f_1(r)}{\partial \alpha} \leq 0 \tag{12}$$

and the result follows.

imately 1, s/he may prefer to reject this offer because s/he knows that if it were accepted it would lead to a number of periods of disagreement. However, this does not affect any of the results.

A.7 Proof of Proposition 6

We first restrict attention to the set $\bar{\mathcal{R}} = [0, \hat{r}_1] \times [0, \hat{r}_2]$. In this region, we know that $f_1(r) \geq r_1$. Consider the region $\bar{\mathcal{R}}_1 = \{r \in \bar{\mathcal{R}} \,|\, \alpha r_i \leq \frac{\delta}{2}\}$. In this region, it is easy to see that $f_i(r) = \frac{\delta}{2}$ and so obviously $\frac{\partial f_i(r)}{\partial \delta} = \frac{1}{2} > 0$. Next consider the set $\bar{\mathcal{R}}_2 = \{r \in \bar{\mathcal{R}} \,|\, r \notin \bar{\mathcal{R}}_1,\, \alpha r \in \bar{\mathcal{R}}_1\}$. Here we have that $f_1(r) = \frac{\delta}{2}\left[1 + f_1(\alpha r) - \max\{f_2(\alpha r), \alpha r_2\}\right]$. Since $\alpha r \in \bar{\mathcal{R}}_1$, we know that $\alpha r_2 \geq \frac{\delta}{2} = f_2(\alpha r)$. Therefore, we have that $f_1(r) = \frac{\delta}{2}\left[1 + f_1(\alpha r) - \alpha r_2\right]$ and so we see that $\frac{\partial f_1(r)}{\partial \delta} = \frac{1}{2}\left[1 + f_1(\alpha r) - \alpha r_2\right] + \frac{\delta}{4} > 0$. One can then easily see that the result follows by induction.

Finally, we can consider points $r \in \Delta$ such that $\alpha r \in \bar{\mathcal{R}}$. It is just as easily seen that $\frac{\partial f_1(r)}{\partial \delta} > 0$. Therefore, we have that $\frac{\partial f_1(r)}{\partial \delta}|_{r=\hat{r}} > 0$ and the result follows.