Improving Retail Profitability by Bundling Vertically Differentiated Products

Dorothée Honhon\(^*\), Xiajun Amy Pan\(^†\)

We consider a retailer managing a category of vertically differentiated goods. The goods can be sold individually, in which case they are referred to as components, and/or in bundles. The retailer chooses the assortment of components and bundles and their selling prices to maximize profit under a capacity constraint. We show that each bundling strategy (pure components, pure bundling or mixed bundling) can be optimal and provide closed form expressions for the optimal prices. When components are in abundant supply, we show that the optimal assortment is independent of the distribution of consumers' valuation for quality and that dominated components, which have a lower quality but higher cost than another component, could be offered in a bundle but not separately. These results no longer hold when component availability is limited. We further demonstrate that bundling vertically differentiated products can greatly improve profits and propose heuristic policies that perform well numerically.

Keywords: Vertical Differentiation, Bundling, Pricing, Assortment

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1 Introduction

We consider a retailer managing a category of vertically differentiated products, i.e., products which
differ with respect to an attribute, for which all consumers agree on the ranking (all else equal). The
differentiating attribute is generally referred to as quality: every customer prefers a product of high
quality to a product of low quality if both products are offered at the same price. In practice the
differentiating attribute can be speed (for processors), memory size (for tablets, USB drives or MP3
players), size (for canned products, household items or cosmetics), freshness (for grocery products), etc.
Vertically differentiated products are offered at different prices and consumers choose which one(s) to
purchase based on how much they value the quality attribute. In contrast, horizontally differentiated
products are such that consumers have different preferences when products are offered at the same
price; color, fat content, sweetness, spice levels are examples of horizontally differentiating attributes.

Bundling, the sale of two or more products for one combined price, is an effective strategy for
retailers to increase profits. For example, a vacation package bundles flight, hotel, and rental car and a
media service package bundles phone, internet, and television. Retailers have since long used bundling
strategies for horizontally differentiated products by combining products of different colors or flavors
into one package, e.g., variety packs with different flavors of potato chips, breakfast cereal or yogurt.
Nowadays some retailers are also experimenting with bundling vertically differentiated products: the
Leonidas chocolate stores in Belgium offer a bundle consisting of a large box of chocolate (which
is typically bought as a gift) and a small box (which is typically for personal consumption), some
supermarkets in China offer a bundle with soon-to-expire milk (for immediate consumption) and milk
with a longer remaining shelf life (for consumption in the coming week), and some US retailers sell a
full size bottle of lotion or shampoo (for daily use at home) packaged together with a travel-size bottle
(for use on the road). From these examples it appears that bundling products of varying quality levels
can be a profitable strategy when consumers purchase the items for different purposes. For example,
consumers may purchase a high quality wine for drinking purposes and a lower quality wine for cooking
purposes and therefore might be interested in purchasing a bundle of the two, or a spa may want to
offer a bundle with an extensive cleansing skin treatment (to be used first) and a quick booster-type
treatment (to be used shortly after the first one to prolong the effects). Our paper shows that a retailer
can indeed significantly increase her profits by offering bundles of vertically differentiated items.
The literature on bundling is extensive, but before us, only Banciu et al. (2010) consider the optimal bundling strategy for vertically differentiated products. This case warrants special attention because of the specific nature of the utility function which governs the choice of consumers, creating a different kind of dependency among the demands of products compared to other models. The analysis by Banciu et al. (2010) focuses on the case of two products with the objective of maximizing sales revenue, that is, they assume a zero variable cost. Their model therefore applies to intangible products such as television commercials and mall rental spaces. In contrast, our analysis focuses on the retail industry. With the current trend of product proliferation in this industry, we need to investigate the optimal bundling and pricing strategy for categories of more than two components. Moreover, retailers typically aim to maximize profit instead of revenue, making it important to include product variable costs in the analysis. Further, considering positive variable costs brings an additional attribute of the products into consideration, namely, the cost-quality ratios, which turns out to be an important determinant of the optimal assortment. In this paper we consider a profit-maximizing retailer who optimizes her bundling, assortment and pricing strategy for a category with \( n \) vertically differentiated items, referred to as components. We consider two cases based on whether the components are available to the retailer in abundant or limited supply, that is, whether or not the retailer optimizes under capacity constraints. Consumers determine what product(s) to purchase by maximizing their utility. While a consumer buys at most one unit of a component, he may buy multiple different components, i.e., a bundle if offered. A bundle may provide more or less utility than the sum of utilities of its individual components depending on the product category. We consider all three possible cases: a super-additive, sub-additive or additive quality relationship among the components.

Current research on bundling distinguishes among three main bundling strategies: pure bundling (offer only bundles), pure components (offer only components) or mixed bundling (offer components and bundles). For only two components with zero cost, Banciu et al. (2010) show that a pure bundling strategy is always optimal when capacity is abundant. In contrast, we show that all three bundling strategies may be optimal with positive component costs. Hence, our results show that the unit component costs play an important role in determining the optimal strategy. Further we show that, when capacity is abundant, the optimal bundling strategy does not depend on the distribution of customer valuation for quality and the optimal assortment does not include any dominated product (we say that a product is dominated if there exists a product with a higher quality level but lower
variable cost); yet, dominated components may be offered in a bundle. Interestingly, both results are no longer valid when capacity is limited. Hence, component scarcity is an important dimension of the problem, and we show that it impacts the optimal bundling strategy in significant, non-monotone ways. We provide an effective algorithm to obtain the optimal assortment as well as closed form expressions for the optimal prices in a very general setting. For a number of special cases, we provide a full characterization of the optimal solution. Finally, we consider a number of heuristic policies, which are simple to calculate and/or implement in practice and test their performance numerically. We show that the profit improvement from adopting bundling can be very significant: in our numerical analysis we obtained an average profit improvement of 87.22% for a product category with 10 components when the quality relationship is super-additive and 9.48% when it is sub-additive. This demonstrates that the benefits of bundling extend to product categories for which consumers value the joint consumption of the components less than the sum of individual consumption. In general, the value of bundling increases with the size of the product category and the strength of the super-additivity relationship.

Our work contributes to the existing literature on bundling by considering vertically differentiated products, positive variable costs, capacity constraints, more than two components, and a non-uniform distribution of consumer valuation for quality. This brand new setting applies particularly well to the retail industry and allows us to generate valuable insights, the main one being that bundling vertically differentiated products can significantly improve profits for a retailer. A strong theoretical contribution of our paper is that we provide closed form expressions for the optimal prices for the mixed bundling strategy, which has been considered analytically intractable in the existing bundling literature except for Banciu et al. (2010) and Bhargava (2013).

The rest of the paper is organized as follows. In Section 2 we review the related literature. In Section 3 we describe the model in detail. The main results are presented in Section 4. In Section 5 we propose a number of heuristic policies, then we discuss their performance and evaluate the value of bundling in Section 6. Section 7 concludes our work and provides directions for future research. All proofs are presented in the Appendix.
2 Literature review

As the popularity of bundling has grown in practice, so has academic research on this topic. Stremersch & Tellis (2002) provide a comprehensive summary of early research in this area and Venkatesh & Mahajan (2009) provide an excellent recent survey of the literature and practice. Early work on bundling, including Stigler (1963), Adams & Yellen (1976), Schmalensee (1984) and McAfee et al. (1989), show that bundling can increase the seller’s profit because consumers’ valuations for the bundle have a smaller variation than their valuations for the individual components. Venkatesh & Kamakura (2003) use a numerical study to identify conditions under which the mixed bundling strategy is optimal.

Due to the complexity of the demand process, most papers do not provide analytical expression for optimal prices given a mixed bundling strategy. One exception is the recent work Bhargava (2013) who provides closed form expressions for a two-component problem. Bakos & Brynjolfsson (2000) and Geng et al. (2005) examine the optimality condition of the pure bundling strategy for a large number of information goods with negligible marginal cost but they do not consider the mixed bundling strategy.

Several studies develop optimization approaches to design and price the bundle. Hanson & Martin (1990) formulate the bundle problem as a mixed integer programming model to determine the optimal bundle prices in order to maximize the profit given the number of consumer segments and consumers’ reservation price in each segment. Bitran & Ferrer (2007) develop a nonlinear mixed integer programming model and a solution approach to determine the optimal composition and price of a bundle meeting specific constraints. Hitt & Chen (2005) and Wu et al. (2008) study the customized bundle pricing strategy for information goods using a non-linear programming approach. In the customized bundling strategy, prices only depend on the number of components in a bundle.

Prasad et al. (2010) consider the bundling problem for products having network externality and characterize the conditions under which the pure bundling, pure components, or mixed bundling strategy with a bundle and one component are optimal. Bhargava (2012) studies the bundling problem in a distribution channel consisting of two manufacturers and one retailer and finds that conflicts in the supply chain make bundling generally less attractive.

So far, only a few papers (Banciu et al. (2010), Bulut et al. (2009), Cao et al. (2013)) have considered the issue of limited component capacity. Also, most papers (e.g., Bakos & Brynjolfsson (1999), Bakos (2013))
Brynjolfsson (2000), Bulut et al. (2009), Wu et al. (2008), Banciu et al. (2010) assume zero variable cost which is justified when dealing with information goods but not retail goods. In our paper we consider limited capacity, positive variable costs and vertical differentiation.

3 Model

Consider a product category with $n$ vertically differentiated alternatives which we refer to as components. The firm may decide to group two or more components into bundles. There are $2^n - n - 1$ possible bundles: $\{1, 2\}, \{1, 3\}, ..., \{1, 2, ..., n\}$. The bundle which includes all the components, i.e., $\{1, ..., n\}$, is called the complete bundle. Let $\mathcal{B}$ denote the set of possible bundles where $\mathcal{B} = \{B : B \subseteq \{1, ..., n\}$ and $|B| > 1\}$. We use the term product to denote either a component or a bundle and use the generic notation $S$, with $S \subseteq \{1, ..., n\}$. There are $N = 2^n - 1$ possible products.

In line with the bundling literature, we assume that consumers cannot benefit from buying more than one unit of the same product and that the resale of products is not permitted or not profitable. We assume that consumers all agree on the quality level of the products; they all view the quality of product $S$ as equal to $Q_S = (\sum_{i \in S} q_i)^\alpha$, where $q_i$ is some intrinsic measure of quality for component $i$ and $\alpha > 0$. Without loss of generality, we assume that $q_1 > q_2 > ... > q_n > 1$ for example, consumers all agree that the quality of consuming components 1 and 2 jointly is $(q_1 + q_2)^\alpha$. The case of $\alpha = 1$ is the strict additivity case wherein the quality levels of the components are individually valued because, for example, they are consumed separately as in the milk example discussed in the introduction. This case has been considered by many authors in the context of non-vertically differentiated goods (see for example Adams & Yellen (1976), Schmalensee (1984), McAfee et al. (1989)). When $\alpha > 1$, we have $(\sum_{i \in S} q_i)^\alpha > \sum_{i \in S} q_i^\alpha$, which means that the quality associated with consuming both components together is greater than the sum of the qualities of the components evaluated separately. In this case we say that quality has a super-additive relationship. This case applies in the wine example discussed in the introduction: the enjoyment of a good bottle of wine can be exacerbated by the taste of a dish featuring a delicious wine-based sauce (and vice versa!). On the other hand, the quality relationship between components is sub-additive when $\alpha < 1$. This would be the case in the spa treatments example.

footnote{1}All of our results continue to hold if the $q_i$ values are lower than 1. We make this assumption to simplify the exposition of some results below.
as the law of diminishing returns probably applies to skin care. Venkatesh & Kamakura (2003) say products are complementary when $\alpha > 1$ and substitutes when $\alpha < 1$ and provide other practical examples. Note that the quality of buying only component $i$ is equal to $q_i^\alpha$. Yet, to simplify the exposition we refer to $q_i$ as the quality level of component $i$.

We assume that there is a fixed capacity of each component. This applies when the physical inventory of the components has been set at the start of the selling period and the retailer no longer has control over it as discussed in Bulut et al. (2009). By assuming a fixed capacity for the components, we are in essence considering price bundling, as opposed to product bundling. In other words, the components are not physically linked to form a bundle (e.g., wrapped together) and no separate inventory of the bundle is kept. However, a price is posted for the different bundles that the firm is offering. Let $u_i$ denote the available inventory/capacity of component $i$. An implicit assumption of our model is that leftover components are disposed of freely at the end of the selling period.

Let $c_i$ denote the variable cost of selling component $i$ for the firm, which corresponds to the operating cost of selling a component. The cost of acquiring the capacity is a sunk cost and therefore ignored. The cost of selling product $S$ is $C_S = \sum_{i \in S} c_i$, i.e., we assume there is no extra cost for bundling the components. By considering positive variable costs, we are able to study the profit-maximizing bundling strategy as opposed to the revenue-maximizing bundling strategy which was previously studied in Banciu et al. (2010). Products are ordered in increasing order of quality so that the $j$-th product $S_j$ has $j$-th highest quality level, $Q_{S_j}$. Let $r_S$ be the price of product $S$. Let $\vec{r} = (r_{S_1}, \ldots, r_{S_N})$ be the vector of product prices.

Customers are heterogenous and are characterized by their willingness to pay for one unit of quality in the product category, or valuation, which is measured by $\theta$. Let $F(\theta) = 1 - (1 - \theta)^b$ with $b > 0$ and $0 \leq \theta \leq 1$ denote the distribution of consumers’ valuations. This distribution is commonly used to model consumer preference (see for example Sundararajan (2004) or Debo et al. (2005)) and counts the uniform distribution as a special case obtained when $b = 1$, as used in Banciu et al. (2010). The utility that a consumer gets from a product is increasing in its quality and decreasing in its price: a consumer with valuation $\theta$ gets utility $U(S, \theta) = \theta (\sum_{i \in S} q_i)^\alpha - r_S$ from product $S$. Without loss of

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2 All of our results in fact hold for a slightly more general case: when $F$ has a linear inverse hazard rate function (the exponential, normal, logistic, chi-squared, Laplace distributions all satisfy this condition). We present our results for this form of $F$ for ease of exposition.

3 All of our results continue to hold for more general utility functions such as $\theta \phi(q_i) - r_j$, where $\phi(\cdot)$ is increasing and concave, and $\xi(\theta)q_j - r_j$, where $\xi(\cdot)$ is increasing and concave.
generality, we assume that the utility of not buying anything is equal to zero. In line with the bundling literature, we assume that the price of each bundle is the cheapest way to buy the components that it includes (see constraint (2) in the model below). Under this assumption, consumers purchase at most one product. A consumer with valuation $\theta$ chooses product $S$ if $U(S, \theta) = \max_{S' \subseteq \{1, \ldots, n\}} U(S', \theta)$ and $U(S, \theta) \geq 0$. Let $P_S$ denote the proportion of consumers who buy product $S$, called the purchase probability of product $S$. We have $P_S = \int_0^1 I(U(S, \theta) = \max_{S' \subseteq \{1, \ldots, n\}} U(S', \theta) \text{ and } U(S, \theta) \geq 0) \, dF(\theta)$, where $I(E)$ is the indicator function for event $E$. The firm’s problem is to find the selling prices $\vec{r}$ which maximize expected profit under the capacity constraints:

$$\begin{align*}
(\mathcal{P}) \quad \mathbb{E} \Pi(\vec{r}^*) &= \max_{\vec{r}} \mathbb{E} \Pi(\vec{r}) \\
\text{such that} & \quad \sum_{S \subseteq \{1, \ldots, n\} : i \in S} P_S \leq u_i \quad \text{for } i = 1, \ldots, n \quad (1) \\
& \quad r_S \leq r_{S_j} + r_{S_k} \quad \text{for all } S \subseteq \{1, \ldots, n\} \text{ and } S_j \cup S_k = S, j \neq k \quad (2)
\end{align*}$$

The firm’s expected profit $\mathbb{E} \Pi$ is equal to $\mathbb{E} \Pi(\vec{r}) = \sum_{S \subseteq \{1, \ldots, n\}} \mu P_S (r_S - C_S)$, where $\mu$ is the expected market size. In what follows we assume that $\mu = 1$ without loss of generality. The firm’s expected profit per product is equal to the profit margin (selling price minus variable cost) multiplied by its purchase probability. Constraints (1) are the capacity constraints for each component. Constraints (2) guarantee that each bundle is the cheapest way to buy all the components which are included in it.

For example, when $n = 2$, the constraint $r_{\{1,2\}} \leq r_{\{1\}} + r_{\{2\}}$ guarantees that consumers who consume both components always buy the bundle instead of components 1 and 2 separately.

Given $\vec{r}$, let $A(\vec{r})$ be the set of products that have a positive purchase probability. We call $A$ the effective assortment of the firm. Products not in $A$ get a zero purchase probability; they are essentially not included in the assortment. Let $A = \{S_{j_1}, S_{j_2}, \ldots, S_{j_m}\}$ where $j_1 < j_2 < \ldots < j_m$. Since the products are numbered in increasing order of quality levels, we have $Q_{S_{j_1}} \leq Q_{S_{j_2}} \leq \ldots \leq Q_{S_{j_m}}$. For all products in $A$ to have a positive purchase probability, it must be true that: $0 < \frac{r_{S_{j_1}}}{Q_{S_{j_1}}}, \frac{r_{S_{j_2}}}{Q_{S_{j_2}}} - \frac{r_{S_{j_1}}}{Q_{S_{j_1}}} < \ldots < \frac{r_{S_{j_m}} - r_{S_{j_{m-1}}}}{Q_{S_{j_m}} - Q_{S_{j_{m-1}}}} < 1$, which implies that $r_{S_{j_1}} < r_{S_{j_2}} < \ldots < r_{S_{j_m}}$ and

$$0 < \frac{r_{S_{j_1}}}{Q_{S_{j_1}}} < \frac{r_{S_{j_2}}}{Q_{S_{j_2}}} < \ldots < \frac{r_{S_{j_m}}}{Q_{S_{j_m}}} < 1,$$  
(3)
and the purchase probabilities are given by:

\[
P_{S_{ji}} = \begin{cases} 
(1 - \frac{r_{S_{ji}}}{Q_{S_{ji}}})^b - (1 - \frac{r_{S_{ji}} - r_{S_{ji-1}}}{Q_{S_{ji}} - Q_{S_{ji-1}}})^b & i = 1 \\
(1 - \frac{r_{S_{ji}} - r_{S_{ji-1}}}{Q_{S_{ji}} - Q_{S_{ji-1}}})^b - (1 - \frac{r_{S_{ji+1}} - r_{S_{ji}}}{Q_{S_{ji+1}} - Q_{S_{ji}}})^b & i = 2, ..., m - 1 \\
(1 - \frac{r_{S_{jm}} - r_{S_{jm-1}}}{Q_{S_{jm}} - Q_{S_{jm-1}}})^b & i = m 
\end{cases}
\]

Figure 1 illustrates an example with two components 1, 2 and the bundle including both components. All the consumers in Group A with \(0 < \theta < \frac{r_{(2)}}{Q_{(2)}}\) get a non-positive utility from each product and thus purchase nothing. Consumers in Group B with \(\frac{r_{(2)}}{Q_{(2)}} \leq \theta < \frac{r_{(1)} - r_{(2)}}{Q_{(1,2)} - Q_{(1)}}\) purchase component 2 because it gives them the highest and positive utility while consumers in Group C with \(\frac{r_{(1)} - r_{(2)}}{Q_{(1,2)} - Q_{(1)}} \leq \theta < 1\) purchase component 1. Finally consumers in Group D with \(\frac{r_{(1,2)} - r_{(1)}}{Q_{(1,2)} - Q_{(1)}} < \theta < 1\) purchase the bundle.

**Figure 1:** Purchase probabilities in an example with 2 components

Based on effective assortment \(A\), we characterize the bundling strategy chosen by the firm. In particular the firm can choose (i) a no bundling or pure components strategy, i.e., \(A \subseteq \{\{1\}, \ldots, \{n\}\}\), (ii) a pure bundling strategy, \(A \subseteq \mathcal{B}\), or (iii) a mixed bundling strategy, i.e., there exists \(\{i\} \in A\) for \(i \in \{1, \ldots, n\}\) and \(B \in A\) for \(B \in \mathcal{B}\). Under the pure components strategy, we distinguish between the full-spectrum pure components strategy, when \(A\) includes all the components and the partial-spectrum pure components strategy when some components are not offered. If \(A\) includes only the complete
bundle, the strategy is referred to as pure complete bundling. Under the mixed bundling strategy, we distinguish between full-spectrum mixed bundling when A includes all the components and all the bundles, and partial-spectrum mixed bundling when A includes only some components and some bundles. In general, the number of possible effective assortments is $2^N - 1$. We summarize our notation in Table 1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>Number of potential components in the category</td>
<td>$S$</td>
<td>product (i.e., component or bundle)</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of potential products in the category,</td>
<td>$Q_S$</td>
<td>Quality level of product S</td>
</tr>
<tr>
<td>$B$</td>
<td>Set of bundles</td>
<td>$C_S$</td>
<td>Cost of product S</td>
</tr>
<tr>
<td>$i$</td>
<td>Component index, $i = 1, \ldots, n$</td>
<td>$r_S$</td>
<td>The price of product S</td>
</tr>
<tr>
<td>$q_i$</td>
<td>Quality level of component $i$</td>
<td>$\bar{r}$</td>
<td>Selling price vector</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Variable cost of component $i$</td>
<td>$\bar{r}^*$</td>
<td>Optimal selling price vector</td>
</tr>
<tr>
<td>$u_i$</td>
<td>Capacity of component $i$</td>
<td>$P_S$</td>
<td>Purchase probability of product S</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Quality relationship parameter</td>
<td>$A(\bar{r})$</td>
<td>Effective assortment</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Consumer valuation</td>
<td>$A^*$</td>
<td>Optimal effective assortment</td>
</tr>
<tr>
<td>$F(\theta)$</td>
<td>Cumulative distribution function of $\theta$</td>
<td>$\Pi$</td>
<td>Profit</td>
</tr>
<tr>
<td>$b$</td>
<td>Parameter for distribution $F$</td>
<td>$\Pi^*$</td>
<td>Optimal profit</td>
</tr>
<tr>
<td>$\mu$</td>
<td>expected market size</td>
<td>$T_i$</td>
<td>Assortment ${1, \ldots, i}$</td>
</tr>
</tbody>
</table>

**Table 1: Notation**

### 4 Optimal bundling strategy

In this section, we investigate the properties of the optimal effective assortment $A^*$ and optimal prices $\bar{r}^*$ and the resulting bundling strategy chosen by the firm. Our first set of results apply to product categories with a super-additive quality relationship.

**Lemma 1.** Suppose the quality relationship is super-additive, i.e., $\alpha > 1$. Any two products in the optimal assortment must have at least one common component, i.e., if $S_{j_1}, S_{j_2} \in A^*$, then $S_{j_1} \cap S_{j_2} \neq \emptyset$.

When $n = 2$, this result implies that the optimal effective assortment cannot include both components 1 and 2: all the consumers who value component 1 higher than 2 buy the bundle over component 1 since the bundle provides them with a higher utility. For general $n$, this implies that some bundling strategies cannot be optimal, as stated in Proposition 1 (proof is omitted).

**Proposition 1.** The full-spectrum mixed bundling and full-spectrum pure components strategies are never optimal when the quality relationship is super-additive, i.e., when $\alpha > 1$.

Lemma 1 and Proposition 1 hold regardless of whether the components are available in abundant or limited quantity. In the next two subsections, we present results specific to these two cases.
4.1 Profit-maximizing strategy when capacity is abundant

In this section we assume that the $n$ components are available in abundant quantities, i.e., $u_i = 1$ for $i = 1, ..., n$, so that problem $(P)$ is solved without the capacity constraints (1). Proposition 2(a) gives necessary and sufficient conditions for the optimal effective assortment $A^*$ and a formula for the optimal prices. Proposition 2(b) show that the optimal assortment is nested when the quality relationship is super-additive.

Proposition 2. 

(a) Let $A^* = \{S_{j_1}, ..., S_{j_m}\}$, where $j_1 < j_2 < ... < j_m$. $A^*$ is optimal if and only if

$$\frac{C_{S_{j_1}}}{Q_{S_{j_1}}} < \frac{C_{S_{j_2}} - C_{S_{j_1}}}{Q_{S_{j_2}} - Q_{S_{j_1}}} < \frac{C_{S_{j_3}} - C_{S_{j_2}}}{Q_{S_{j_3}} - Q_{S_{j_2}}} < ... < \frac{C_{S_{j_m}} - C_{S_{j_{m-1}}}}{Q_{S_{j_m}} - Q_{S_{j_{m-1}}}} < 1 \quad (4)$$

$$\frac{C_{S_{j_1}}}{Q_{S_{j_1}}} < \frac{C_{S_{j_2}}}{Q_{S_{j_2}}} < ... < \frac{C_{S_{j_m}}}{Q_{S_{j_m}}} < 1 \quad (5)$$

$$\frac{C_{S_k} - C_{S_{j_{i-1}}}}{Q_{S_k} - Q_{S_{j_{i-1}}}} \geq \frac{C_{S_{j_i}} - C_{S_k}}{Q_{S_{j_i}} - Q_{S_k}}, \text{ for } i = 1, ..., m \text{ and } k = j_{i-1} + 1, ..., j_i - 1, \quad (6)$$

$$\frac{C_{S_k} - C_{S_{j_m}}}{Q_{S_k} - Q_{S_{j_m}}} \geq 1, \text{ for } k = j_m + 1, ..., N. \quad (7)$$

where $C_{S_{j_0}} = Q_{S_{j_0}} = 0$. Moreover, the optimal effective assortment $A^*$ is unique and does not depend on the consumer valuation distribution $F$. The optimal product prices for $S \in A^*$ are such that

$$r^*_{S} = \frac{bC_S + QS}{1 + b}. \quad (8)$$

For $S \notin A^*$, $r^*_{S}$ is given by (8) when $\alpha < 1$ and $+\infty$ otherwise.

(b) When $\alpha > 1$, if $S, S' \in A^*$ such that $Q_S < Q_{S'}$, then $S \subset S'$. In other words, the products included in the optimal effective assortment are nested sets.

Conditions (4) and (5) imply that the cost-quality ratios of products in the optimal effective assortment $A^*$ are strictly increasing in the quality levels. The optimality conditions (4) to (7) lead to the unique assortment $A^*$. Once $A^*$ is known, equations (8) provide a method to compute the optimal prices $r^*$. Note that the pricing formula from (8) is intuitive since the selling price is increasing in

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4These prices are calculated so as to satisfy constraints (2). In practice, products which are not offered by the retailer simply do not have a selling price.
both the quality level and the variable cost of the product. Further, as \( b \) increases, which means that all consumers tend to value the quality of the product less, the relative impact of the quality on prices decreases.

The result that the optimal effective assortment \( A^* \) does not depend on the consumer valuation distributions \( F \) is somewhat counterintuitive. This means that the optimal bundling strategy is not a function of consumers’ valuation for quality. However, the optimal prices do depend on it. Also note that the optimal price for product \( S \) only depends on the cost and quality for product \( S \), not on those of the other products, which provides useful pricing guidance for practitioners: if a firm decides to introduce a new product, it can do so without changing the prices of the existing products.

By Proposition 2(b), the products in the optimal assortment are nested sets when the quality relationship is super-additive, which means that buying a bundle is the only way to buy all the components which are included in it. For example, when \( n = 2 \), buying the \( \{1, 2\} \) bundle (provided it is offered) is the only way to buy both components because at least one of the two will not be offered separately.

In contrast, when the quality relationship is sub-additive, consumers may be able to “build their own bundle” by purchasing components 1 and 2 separately. However they will never do so since the bundles are always offered at a positive discount compared to any such basket purchase, as stated in Lemma 2. This is because consumers do not get extra utility from consuming components together, therefore the retailer has to offer a positive discount on bundles in order to make the bundle appealing.

**Lemma 2.** For products with sub-additive relationship, i.e., \( \alpha < 1 \), the optimal selling price of every bundle \( S \in \mathcal{B} \) is such that \( r^*_S < \sum_{j=1}^{k} r^*_{S_j} \), for all \( S_1, \ldots, S_k \subset \{1, \ldots, n\} \) such that \( S_1 \cup \ldots \cup S_k = S \) and \( S_i \cap S_j = \emptyset \) for \( i, j \in \{1, \ldots, k\}, i \neq j \) i.e., the bundles are sold at a strictly positive discount compared to the sum of the prices of the products that are included in it.

We say that product \( S \) is *dominated* if there exists a product \( S' \neq S \) with higher quality and lower cost, i.e., \( C_{S'} \leq C_S \) and \( Q_{S'} \geq Q_S \) and at least one of the two inequalities is strict; otherwise it is *non-dominated*. In Proposition 3 we show that dominated products are not included in the optimal assortment.

**Proposition 3.** The optimal prices \( \mathbf{r}^* \) are such that the products with positive purchase probabilities are non-dominated.
However, while a component may be (individually) dominated, it is possible that it is offered as part of a bundle in the optimal assortment. This result implies that the set of components which can be purchased by the customers may expand when the retailer considers the option of bundling.

Next, we discuss how to obtain the optimal effective assortment $A^*$. When $\alpha \leq 1$ the optimal effective assortment $A^*$ is simply the set of products which have a positive purchase probability when the prices of all products are set equal to (8). When $\alpha > 1$, the optimal effective assortment and prices can be obtained using Algorithm 1, which is adapted from Pan & Honhon (2012), and is polynomial in the number of products (non-dominated) $N$.

**Algorithm 1** Solve for the optimal effective assortment and prices

**Step 0:** $A^* = \emptyset$, $i = 0$.

**Step 1:**

if $i < N$ then

for $j := N$ down to $i + 1$ do

if $\frac{C_{S_j} - C_{S_k}}{Q_{S_j} - Q_{S_k}} < 1$ AND $\frac{C_{S_k} - C_{S_j}}{Q_{S_k} - Q_{S_j}} > \frac{C_{S_j} - C_{S_k}}{Q_{S_j} - Q_{S_k}}$ for $k = i + 1, \ldots, j - 1$ then

$A^* := A^* \cup \{j\}$, $i := j$ and back to Step 1.

end if

end for

end if

**Step 2:** Use Equations (8) to obtain $\bar{r}^*$.

Example 1 illustrates the results from Proposition 2.

**Example 1.** A firm can choose from three vertically differentiated products with $\bar{c} = (10, 3, 5)$, $\bar{q} = (30, 20, 12)$ and $\alpha = 1.2$. The optimal bundling strategy, expected profit, assortment, price, and purchase probability for different $b$ values are shown in Table 2.

<table>
<thead>
<tr>
<th>$b$</th>
<th>Partial-spectrum Bundling</th>
<th>Partial-spectrum Bundling</th>
<th>Partial-spectrum Bundling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Profit</td>
<td>$E\Pi^* = 44.4$</td>
<td>$E\Pi^* = 27$</td>
<td>$E\Pi^* = 14$</td>
</tr>
<tr>
<td>Optimal Effective Assortment</td>
<td>${2}, {1, 2}, {1, 2, 3}$</td>
<td>${2}, {1, 2}, {1, 2, 3}$</td>
<td>${2}, {1, 2}, {1, 2, 3}$</td>
</tr>
<tr>
<td>Optimal Price</td>
<td>$(25.3, 77.2, 100.4)$</td>
<td>$(19.7, 61.2, 79.8)$</td>
<td>$(14.1, 45.1, 59.2)$</td>
</tr>
<tr>
<td>Purchase Probability</td>
<td>$(1.7%, 0.6%, 53.1%)$</td>
<td>$(2.7%, 0.9%, 42.2%)$</td>
<td>$(4.3%, 1.4%, 31.7%)$</td>
</tr>
</tbody>
</table>

Table 2: Bundling strategies, optimal assortments, price vectors, expected profit, and purchase probability as a function of $b$ in Example 1.

We observe that the bundling strategy and the optimal assortment do not vary with $b$, i.e., the consumer variation distribution $F$, which is consistent with Proposition 2. The optimal effective assortment $A^*$ and thus the optimal bundling strategy only depend on the cost-quality ratios of the products. However, the optimal prices, the purchase probabilities, and thus the optimal expected profit do vary.
with the consumer valuation distribution $F$.

In practice the distribution of consumer valuations may change over time. When a new product is introduced, most consumers are likely to have low valuation for quality, as they are unsure of the product benefits. As the market matures, the distribution likely shifts towards higher valuations (Sundararajan (2004)). A gradual decrease in the parameter $b$ of the consumer distribution $F$ reflects the market evolution of products.

**Lemma 3.** As the market matures, i.e., as $b$ decreases, the optimal effective assortment remains unchanged but selling prices and total market share increase.

This result shows that in the early stages of a product life cycle, the proportion of consumers buying nothing is high and a low bundling price is a good strategy to penetrate the market. As the market matures, the firm can gain more profits by increasing the product prices without changing the assortment. Moreover, the total market share increases over time.

In some industries, the variable costs does not vary across components. For example, Wu et al. (2008) report that the variable cost of a song sold on iTunes is 50 cents for every song. We explore this special case in the next subsection as well as the special case of the two-component problem, both of which have received a lot of attention in the bundling literature.

4.1.1 Identical component costs

When component costs are identical, i.e., $c_i = c$ for $i = 1, ..., n$, Proposition 4 shows that the products in the optimal effective assortment all include a certain number of the highest quality components. Formally let $T_i = \{1, ..., j\}$ with $i \in \{1, ..., n\}$.

**Proposition 4.** When $c_i = c$ for $j = 1, ..., n$, the optimal prices $\bar{r}^*$ are such that, if $S \in A^*$ then $S = T_i$ for $i \in \{1, ..., n\}$.

This result follows directly from Proposition 3 since sets other than $T_i$, $i = 1, ..., n$ are dominated. This result is reminiscent of the result from Van Ryzin & Mahajan (1999) in assortment planning with horizontally differentiated products: when consumer preferences are modeled using the Multinomial
Logit Model, the optimal assortment contains a certain number of the most popular variants. Note that, in our setting, the optimal effective assortment may not include all the \( T_i \) sets; for example, we could have \( A^* = \{\{1\}, \{1, 2, 3\}\} \).

It is interesting to know when the optimal effective assortment includes the complete bundle \( T_n = \{1, 2, ..., n\} \); Lemma 4 provides the necessary and sufficient condition.

**Lemma 4.** The optimal set \( A^* \) includes the complete bundle \( T_n \), i.e., \( T_n \in A^* \), if and only if

\[
\max_{i=0, ..., n-1} \frac{(n - i)c}{Q_{T_n} - Q_{T_i}} < 1 \tag{9}
\]

In (9), \( \frac{(n - i)c}{Q_{T_n} - Q_{T_i}} \) can be interpreted as the incremental-cost-to-incremental-quality ratio when comparing products \( T_i \) with \( T_n \). When this value is less than 1 for all \( i \), offering the complete bundle \( T_n \) is profitable because the incremental quality it brings is greater than the incremental cost compared to any product \( T_i \), \( i = 1, ..., n - 1 \).

Banciu et al. (2010) show that pure bundling is optimal when the component capacities are not binding for the revenue maximization problem (i.e., assuming \( c = 0 \)) with \( n = 2 \). Lemma 5 indicates that their result continues to hold for \( n > 2 \) components.

**Lemma 5.** When \( c = 0 \), the pure complete bundling strategy is optimal, i.e., \( A^* = \{T_n\} \).

Let \( \Delta Q_{T_i} = Q_{T_i} - Q_{T_{i-1}} = \left( \sum_{j=1}^{i} q_j \right)^{\alpha} - \left( \sum_{j=1}^{i-1} q_j \right)^{\alpha} \) denote the quality differential between products \( T_i \) and \( T_{i-1} \), for \( i = 1, ..., n \). We prove in Appendix A (Lemma 10) that there exist two threshold values \( \underline{\alpha} \) and \( \overline{\alpha} \) such that \( \Delta Q_{T_1} > \Delta Q_{T_2} > ... > \Delta Q_{T_n} \) for \( \alpha < \underline{\alpha} \) and \( \Delta Q_{T_1} < \Delta Q_{T_2} < ... < \Delta Q_{T_n} \) for \( \alpha > \overline{\alpha} \). Further, we have \( 1 \leq \underline{\alpha} \leq \overline{\alpha} \). Proposition 5 provides a full characterization of the optimal effective assortment when \( c > 0 \).

**Proposition 5.** For a positive component cost, i.e., \( c > 0 \).

(i) For products with sub-additive or weakly super-additive quality relationship, i.e., such that \( \alpha \leq \underline{\alpha} \), the optimal effective assortment is \( A^* = \{T_1, T_2, ..., T_{i^*}\} \) where \( i^* \) is the largest integer in \( \{1, ..., n\} \) such that \( c < \Delta Q_{T_{i^*}} \). If \( c > \Delta Q_{T_1} \), then it is optimal to offer nothing.

(ii) For products with strongly super-additive quality relationship, i.e., such that \( \alpha \geq \overline{\alpha} \), it is optimal
to offer only the complete bundle, i.e., \( A^* = \{ T_n \} \), when \( nc < Q_{T_n} \); otherwise, offer nothing.

It follows that, for \( \alpha \leq \bar{\alpha} \), the pure components strategy is optimal when \( i^* = 1 \) and the mixed bundling strategy is optimal when \( i^* > 1 \). In contrast, the pure complete bundling strategy is optimal when \( \alpha > \bar{\alpha} \).

Figure 2 shows how the optimal bundling strategy changes with \( \alpha \) and \( c \) (PC, PB, MB stands for Pure Components, Pure Bundling, and Mixed Bundling strategy respectively) in a particular example. We see that, for average values of \( c \) and a strongly sub-additive quality relationship, the firm adopts a pure components strategy, with only component 1. As \( \alpha \) increases up to \( \bar{\alpha} \), the firm switches to a partial-spectrum mixed bundling strategy. When \( \alpha \) increases further but lower than \( \bar{\alpha} \), the optimal bundling strategy is pure bundling offering \( \{1, 2\} \), \( \{1, 2, 3\} \). As the quality relationship becomes strongly super-additive with \( \alpha \geq \bar{\alpha} \), the pure bundling strategy offering the complete bundle only gives the highest profit.

Figure 2: Bundling strategy as a function of \( \alpha \) and \( c \) when \( n = 3 \), \( \bar{q} = (30, 20, 12) \) and \( b = 1 \). In this case, \( \underline{\alpha} = 1.3 \) and \( \bar{\alpha} = 1.9 \).

Lemma 6 shows how the bundling strategy, the product prices, and the total market share change with the component cost: when the component cost goes up, the retailer offers fewer products and increases the selling price(s) of the product(s) which remain in the assortment. As a result, the market share goes down.

Lemma 6. For positive cost \( c > 0 \),

(i) for products with sub-additive or weakly super-additive quality relationship, i.e., such that \( \alpha \leq \underline{\alpha} \),

the size of the effective assortment and total market share decrease and the prices of offered
products increase as the variable component cost $c$ increases up to $Q_{T_1}$ (after which it becomes optimal to offer nothing),

(ii) For products with strongly super-additive quality relationship, i.e., such that $\alpha \geq \overline{\alpha}$, the optimal bundle price increases and the total market share decreases as the variable component cost $c$ increases up to $Q_{T_n}$ (after which it becomes optimal to offer nothing).

When $\alpha$ increases, consumers value the products more and therefore the retailer charges higher prices to extract more consumer surplus, as stated in the next lemma.

**Lemma 7.** The optimal prices of products are monotonically increasing in $\alpha$.

### 4.1.2 The two-component problem

In this section, we consider the special case where there are only two components, i.e., $n = 2$. In this case, there are seven possible bundling strategies as shown in Table 3.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Effective assortment $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pure bundling (PB)</td>
<td>${{1,2}}$</td>
</tr>
<tr>
<td>full-spectrum pure components (PC12)</td>
<td>${{1}}$</td>
</tr>
<tr>
<td>partial-spectrum pure components with only component 1 (PC1)</td>
<td>${{2}}$</td>
</tr>
<tr>
<td>partial-spectrum pure components with only component 2 (PC2)</td>
<td>${{1}, {2}}, {1,2}$</td>
</tr>
<tr>
<td>full-spectrum mixed bundling (MB12)</td>
<td>${{1}, {2}, {1,2}}$</td>
</tr>
<tr>
<td>partial-spectrum mixed bundling with component 1 (MB1)</td>
<td>${{1}, {1,2}}$</td>
</tr>
<tr>
<td>partial-spectrum mixed bundling with component 2 (MB2)</td>
<td>${{2}, {1,2}}$</td>
</tr>
</tbody>
</table>

**Table 3:** Possible bundling strategies with $n = 2$.

Let $\beta = \frac{c_1}{c_2}$ be a parameter which captures the relative difference in costs between the two components. If $\beta \leq 1$, product 1 has a higher quality but cheaper cost than product 2, so that product 2 is dominated. This case exists in some industries where creating an inferior version of a product is costly (see Deneckere & McAfee [1996]). We provide a full characterization of the optimal solution as a function of $c_1, c_2, q_1, q_2, \alpha$ and $\beta$.

**Proposition 6.** Table 4 presents a full characterization of the optimal solution when $n = 2$.

Table 4 shows that, for products with a sub-additive quality relationship, the firm never adopts a *pure bundling* (PB) strategy but all other strategies may be optimal. This is because the quality
of a bundle is less than the sum of qualities of its components; therefore, most consumers do not benefit from consuming both components and the firm has no incentive to offer only the bundle. On the other hand, for products with a super-additive quality relationship, the firm will not adopt a full-spectrum pure components (PC12) or a full-spectrum mixed bundling (MB12) strategy. This result follows directly from Proposition 1.

Figure 3 illustrates how the optimal bundling strategy changes with $\alpha$ and $\beta$ in a particular example. Notice that component 2 is never offered separately when it is dominated by component 1, that is when $\beta \leq 1$, which is consistent with Proposition 3. As the quality relationship parameter $\alpha$ increases, the firm switches from a pure components bundling strategy to a mixed bundling strategy, then to a pure bundling strategy.

Banciu et al. (2010) study the two-component case with vertically differentiated products when costs are negligible ($c = 0$). Unlike us they find that the optimal strategy is always pure bundling when component capacities are not binding. In footnote 8 on page 2209 they conjecture that positive component cost would not change the general thrust of their findings. Our results show that their conjecture is not correct: the full array of bundling strategies can be optimal when the component cost is positive.
McAfee et al. (1989) consider the bundling problem of two components with different component costs. They show that the mixed bundling strategy weakly dominates the pure bundling strategy and the mixed bundling strategy is optimal for all independently distributed reservation values (which corresponds to our $\alpha = 1$ case) and could be optimal for negatively and positively related distribution of reservation values (corresponding to our $\alpha > 1$ and $\alpha < 1$ cases respectively). They provide conditions under which mixed bundling is optimal but do not specify how to optimally price the components and/or the bundle. In contrast, we provide optimal conditions for each possible bundling strategy and the optimal pricing scheme. Unlike them we find that pure bundling strategy could be optimal for products with super-additive quality relationship and pure components strategy could be optimal for products with additive quality relationship. This suggests that profit-maximizing retailers cannot use the results obtained in the existing literature given the special nature of the relationship between components.

When the two components have identical costs, i.e., $c_1 = c_2 = c$ and we have $\alpha = \overline{\alpha} = \frac{\ln 2}{\ln(1 + q_2/q_1)}$ and the optimal solution is given in Proposition 7.

**Proposition 7.** Let $\overline{\alpha} = \frac{\ln 2}{\ln(1 + q_2/q_1)} > 1$. Table 5 provides a full characterization of the optimal solution when $n = 2$ and $c_1 = c_2 = c$.

Note that product 2 is a dominated product therefore it is never optimal to include it in the optimal effective assortment. Figure 4 represents the optimal bundling strategy in a specific example. We see that as $\alpha$ increases, the optimal bundling strategy switches from pure components to mixed bundling to pure bundling and the first switch tends to occur for larger values of $\alpha$ as $c$ increases.
Venkatesh & Kamakura (2003) also study the special case of two components with identical variable cost when consumers have (independently) uniformly distributed reservation prices for the components. Through simulation, they find that all three bundling strategies can be optimal as the component cost and ‘degree of contingency’ (which corresponds to our \( \alpha \) parameter as it measures the substitute/complement relationship between the components) vary. Comparing their Figure 3b to our Figure 4, we see some similarities, e.g., the optimality of the pure components strategy in the case of high variable cost and low \( \alpha \). However there are some important differences. In particular, our results indicate that a pure bundling strategy is always optimal when \( c = 0 \), while Venkatesh & Kamakura (2003) show that mixed bundling can also be optimal. Moreover, Venkatesh & Kamakura (2003) find that the firm may switch from a pure bundling to a pure components strategy as \( \alpha \) increases. In contrast, with vertically differentiated products, we show that the bundle always becomes more appealing to the consumers when the quality relationship between components becomes more super-additive.
4.2 Profit-maximizing strategy when capacity is binding

In this section we assume that the components are available in limited quantities. In practice, new models of very popular products such as cell phones, tablets or game consoles may originally be available to the retailer in limited quantities due to rationing by the manufacturer. Over time, the availability of these products is likely to increase as supply matches demand more closely. Alternatively, supply chain disruptions can make some products temporarily hard to source. In this section, we explore how the scarcity of the components may impact the optimal bundling strategy for vertically differentiated products.

In general, for a problem with $n$ products, problem $P$ has $n + C^2_{n-1}$ constraints. In theory, it can be solved using the Karush-Kuhn-Tucker (KTT) conditions but the number of cases to consider grows very quickly with $n$. In what follows, we focus on the two-component problem, i.e., $n = 2$. The following example illustrates some properties of the optimal solution.

**Example 2.** A firm can choose from two vertically differentiated products with $\vec{c} = (0.9, 1.2)$, $\vec{q} = (5, 4.5)$, $\alpha = 0.8$ and $u = (0.1, 0.2)$. The optimal bundling strategy, expected profit, assortment, price, and purchase probability for different $b$ values are shown in Table 2.

<table>
<thead>
<tr>
<th>$b$</th>
<th>Bundling Strategy</th>
<th>Optimal Effective Assortment</th>
<th>Optimal Price</th>
<th>Purchase Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
<td>full spectrum pure components</td>
<td>${{1}, {2}}$</td>
<td>$(3.32, 3.03)$</td>
<td>$(10%, 20%)$</td>
</tr>
<tr>
<td>$1$</td>
<td>full-spectrum mixed bundling</td>
<td>${{1}, {2}, {1, 2}}$</td>
<td>$(2.91, 2.65, 5.11)$</td>
<td>$(1%, 10%, 10%)$</td>
</tr>
<tr>
<td>$2$</td>
<td>partial-spectrum mixed bundling</td>
<td>${{2}, {1, 2}}$</td>
<td>$(1.91, 3.77)$</td>
<td>$(8%, 10%)$</td>
</tr>
</tbody>
</table>

**Table 6:** Bundling strategies, optimal assortments, price vectors, expected profit, and purchasing probability as a function of $b$ in Example 1.

First we observe that, contrary to the case when capacity is not binding, the bundling strategy and the optimal assortment vary with $b$, i.e., with the consumer variation distribution $F$. Also we see that the optimal effective assortment may include dominated products (component 2). This shows that component scarcity plays an important role in determining the optimal bundling strategy.

For ease of exposition, we use the uniform distribution for consumer valuations in the rest of this section, i.e., $b = 1$. Table 2 in Appendix B gives the expressions for the optimal prices for each optimal bundling strategy based on which constraint(s) is(are) binding. Some important necessary conditions
for the optimality of each case are also provided. From this table, we can see that the same optimal bundling strategy can arise with different selling prices, depending on which capacity constraints are binding. When both components are in short supply, the optimal prices do not depend on the costs but on the quality and the capacity of each component. However, the shadow prices do depend on the cost parameters as shown in Table 13. Having an additional unit of a scarce resource can potentially generate additional sales but may also decrease the selling price. Shadow prices indicate how much retailers should be willing to pay for an additional unit of a scarce component.

Proposition 8 states how the optimal bundling strategy changes when the supply of components becomes scarce.

**Proposition 8.** Assume $F$ is uniform, i.e., $b = 1$. Suppose the supply of component switches from being abundant, i.e., $u_1 = u_2 = 1$, to being limited, i.e., $u_1, u_2 \in (0, 1)$.

(i) If the optimal bundling strategy for abundant supply was MB12, the optimal bundling strategy for limited supply can only be MB12, PC12, MB1 or MB2.

(ii) If the optimal bundling strategy for abundant supply was PB, the optimal bundling strategy for limited supply can only be PB, MB1 or MB2.

(iii) If the optimal bundling strategy for abundant supply was MB1 or MB2, the optimal bundling strategy for limited supply can only be MB12, MB1, MB2, PB or PC12.

(iv) If the optimal bundling strategy for abundant supply was PC12, it remains the optimal bundling strategy when supply is limited. If the optimal bundling strategy for abundant supply was PC1, the optimal bundling strategy changes to PC12 when the capacity of component 1 is relatively tight compared to component 1.

Proposition 8(i) implies that component scarcity can make bundling less attractive so that the retailer switches to a pure components strategy. As shown in Figure 5, the retailer uses the full-spectrum bundling strategy (MB12) when the components are in abundant supply for a product category with sub-additive quality relationship. When one component is in short supply, the retailer offers the bundle along with the other component only (MB1 or MB2). When both components are scarce, the optimal strategy becomes pure components (PC12).

---

3The full set of necessary and sufficient conditions is quite involved and is available upon request.
Proposition 8(ii) implies that component scarcity may make bundling less attractive but the retailer never switches to a pure components strategy. Figure 6 shows how a retailer changes her bundling strategy for a product category with super-additive quality relationship. If the supply of one component is relatively tight compared to the other, the retailer adopts partial-spectrum bundling strategy (MB1 or MB2) by offering the bundle with the component in abundant supply; otherwise, the pure bundling strategy remains optimal.

Comparing Figures 5 and 6 with Figures 2(a) and (b) in Banciu et al. (2010), we see that the positive variable cost expands the region of full-spectrum mixed bundling for products with sub-additive relationship and the region of pure bundling for products with super-additive relationship. Therefore, the positive variable cost plays an important role in the bundling decision.

The case from Proposition 8(iii) is illustrated in Figures 7 and 8. Interestingly we see on Figure 7 that the optimal bundling strategy can switch from mixed bundling to pure bundling then back to mixed bundling as component capacity decreases. This suggests that the impact of component scarcity is not one-directional: it can lead to more or less bundling depending on the situation.

Proposition 8(iv) is illustrated in Figure 9 and shows that component scarcity cannot make the retailer switch from a no-bundling (i.e., pure components) strategy to a bundling strategy. We see that partial spectrum pure components (PC1) is optimal until the supply of component 1 becomes really tight, then full spectrum pure components strategy becomes optimal (PC12).
In summary, our results show that all possible bundling strategies can arise when the supply of components is limited and that the optimal strategy changes with component availability. As the supply of components decreases, the optimal bundling strategy may switch from i) mixed bundling to pure bundling; (ii) pure bundling to mixed bundling; (iii) mixed bundling to pure components, which shows that the impact of component scarcity on the bundling strategy is not one-directional. Interestingly, the optimal bundling strategy never switches from pure bundling to pure components.
5 Heuristic strategies

As discussed in the introduction, retailers currently do not generally bundle vertically differentiated goods. In this section we examine how much extra profit could be gained through this strategy. In particular, we assume the firm ignores the fact that consumers may purchase more than one component when determining selling prices, then we calculate the optimality gap of such policy, which we call the “ignore bundling (IB)” strategy. We call the optimality gap of this strategy the value of bundling.

The optimal pricing and bundling strategy obtained in the previous section could be difficult to calculate and/or to implement in practice if the number of products in the optimal effective assortment is too large (it can be up to $2^n - 1$). Therefore, we propose the following heuristic pricing strategies: the “Components Only (CO)” strategy, where the firm sets selling prices only for individual components while taking the consumers’ bundling behavior into consideration, the “Cost-Quality pricing (CQ)” heuristic, where the firm sets the price of every possible product using the simple pricing formula in [8], and the “Complete Bundling Only (BO)” strategy, where the retailer offers only the complete bundle in his assortment. Interestingly, some of these heuristics achieve the optimal profit under certain conditions, which we present below. In Section 6 we report their numerical performance.

**Ignore Bundling heuristic** Under the IB heuristic, the firm only sets prices for (some of) the $n$ components, denoted $r_1^{IB}, ..., r_n^{IB}$, assuming that consumers will purchase at most one of these components. The problem is:

\[(P_{IB}) \quad \mathbb{E} \Pi^{IB}(r_1^{IB}, ..., r_n^{IB}) = \max_{r_1^{IB}, ..., r_n^{IB}} \mathbb{E} \Pi^{IB}(r_1^{IB}, ..., r_n^{IB})\]

such that $P_i^{IB} \leq u_i$ for $i = 1, ..., n$

where $\mathbb{E} \Pi^{IB}(r_1^{IB}, ..., r_n^{IB}) = \sum_{i=1}^{n} P_i^{IB}(r_i - c_i)$ and $P_i^{IB}$ is incorrectly calculated as

\[P_i^{IB} = \int_0^1 I \left( \theta q_i^\alpha - r_i^{IB} = \max_{j=1, ..., n} (\theta q_j^\alpha - r_j^{IB}) \text{ and } \theta q_i^\alpha - r_i^{IB} \geq 0 \right) dF(\theta). \quad (10)\]

However, given prices $r_1^{IB}, ..., r_n^{IB}$, consumers may buy multiple components. They will buy the set
\[ S \subseteq \{1, \ldots, n\} \text{ which maximizes their utility and the correct purchase probability for } S \text{ is given by:} \]

\[
P_S = \int_0^1 I \left( \theta Q_S - \sum_{i \in S} r_i^{IB} = \max_{S' \subseteq \{1, \ldots, n\}} \left( \theta Q_{S'} - \sum_{i \in S'} r_i^{IB} \right) \text{ and } \theta Q_S - \sum_{i \in S} r_i^{IB} \geq 0 \right) dF(\theta) \quad (11)\]

When capacity is abundant, i.e., \( u_i = 1 \) for \( i = 1, \ldots, n \), the firm’s expected profit is calculated as \( \mathbb{E}\Pi(r_1^{IB}, \ldots, r_n^{IB}) = \sum_{S \subseteq \{1, \ldots, n\}} P_S (\sum_{i \in S} r_i^{IB} - C_S) = \sum_{i=1}^n (\sum_{S:i \in S} P_S) (r_i^{IB} - c_i) \). In this case, the problem can be solved using the zero Fixed Cost Algorithm in Pan & Honhon (2012) and the selling price of product \( i \) is equal to \( r_i^{IB} = \frac{b_i + q^*_i}{b + 1} \) if it is offered and \(+\infty\) otherwise. If capacity is limited, because the firm has calculated prices \( r_1^{IB}, \ldots, r_n^{IB} \) based on (10) instead of (11), the demand for component \( i \), which is equal to \( \sum_{S:i \in S} P_S \), may exceed its capacity \( u_i \). In this case, we assume the firm sells \( u_i \). Hence, we have \( \mathbb{E}\Pi(r_1^{IB}, \ldots, r_n^{IB}) = \sum_{i=1}^n \min \{ \{\sum_{S:i \in S} P_S, u_i\} \} (r_i^{IB} - c_i) \). This is consistent with the assumption of assortment-based substitution from the assortment planning literature, i.e., consumers do not substitute in the event of a stock out of their most preferred item. Table 14 in Appendix B provides the full characterization of the solution for the limited capacity case when \( n = 2 \).

**Components only heuristic** Under the CO heuristic, the firm only sets selling prices for (some of) the \( n \) individual components, denoted \( r_1^{CO}, \ldots, r_n^{CO} \), taking consumers’ bundling behavior into consideration. In other words, the firm accounts for the possibility that consumers may buy multiple components by “building their own bundle”. The total price of set \( S \subseteq \{1, \ldots, n\} \) is \( \sum_{i \in S} r_i^{CO} \). The problem becomes:

\[
(\mathcal{P}_{CO}) \quad \mathbb{E}\Pi(r_1^{CO}, \ldots, r_n^{CO}) = \max_{r_1^{CO}, \ldots, r_n^{CO}} \mathbb{E}\Pi(r_1^{CO}, \ldots, r_n^{CO}) \\
\text{such that } \sum_{S \subseteq \{1, \ldots, n\}; j \in S} P_S \leq u_i \quad \text{for } i = 1, \ldots, n

where \( P_S \) is given by

\[
P_S = \int_0^1 I \left( \theta Q_S - \sum_{i \in S} r_i^{CO} = \max_{S' \subseteq \{1, \ldots, n\}} \left( \theta Q_{S'} - \sum_{i \in S'} r_i^{CO} \right) \text{ and } \theta Q_S - \sum_{i \in S} r_i^{CO} \geq 0 \right) dF(\theta)
\]

and the firm’s expected profit \( \mathbb{E}\Pi \) is equal to \( \mathbb{E}\Pi(r_1^{CO}, \ldots, r_n^{CO}) = \sum_{S \subseteq \{1, \ldots, n\}} P_S (\sum_{i \in S} r_i^{CO} - C_S) = \sum_{i=1}^n (\sum_{S:i \in S} P_S) (r_i^{CO} - c_i) \).
Finding the prices under the CO heuristic can be done using KKT conditions. Table 15 in Appendix B provides the expressions for the selling prices given by the CO heuristic for a two-component problem. When $n > 2$, $\alpha \geq 1$ and capacity is abundant, we are able to exploit the nested property of the optimal effective assortment established by Proposition 2(b) to calculate the component prices (see (16) in the proof of Proposition 10). For other cases, calculating the CO heuristic prices is very time consuming.

Cost-Quality pricing heuristic Under the CQ heuristic, the firm uses (8), i.e., $r^CQ_S = \frac{bC_S + Q_S}{b+1}$, to set the prices of all the possible products. As previously argued, this pricing formula is intuitive as the selling price is increasing in both the variable cost and quality of the product. However it requires providing a selling price for up to $2^n - 1$ different products, which can be impractical when $n$ is large. Consumers observe these prices and decide which components or combination of components they wish to purchase. To do so, for each set $S \subseteq \{1,\ldots,n\}$, they calculate the cheapest way to purchase the components in $S$, denoted $r^CQ_S$, which is equal to

$$r^CQ_S = \begin{cases} r^CQ_S & \text{if } |S| = 1 \\ \min_{S_j, S_k \subseteq \{1,\ldots,n\}: S_j \cup S_k = S, j \neq k} r^CQ_S + r^CQ_{S_k} & \text{if } |S| > 1 \end{cases}$$

Then the purchase probability of each product is given by

$$P_S = \int_0^1 I \left( \theta Q_S - r^CQ_S = \max_{S' \subseteq \{1,\ldots,n\}} (\theta Q_{S'} - r^CQ_{S'}) \text{ and } \theta Q_S - r^CQ_S \geq 0 \right) dF(\theta)$$

Note that it is likely that some of the products will receive a zero purchase probability. The firm’s expected profit is calculated as $\mathbb{E}\Pi(r^CQ_1,\ldots,r^CQ_n) = \sum_{S \subseteq \{1,\ldots,n\}} P_S \left( r^CQ_S - C_S \right)$.

When capacity is limited, it is possible that the chosen prices lead to a demand for components which is higher than their capacity. In this case, we propose the following rationing mechanism: products are allocated to consumers in decreasing order of their profit margin $(r^CQ_S - C_S)$ for the retailer. This allocation scheme provides an upper bound on the expected profit so that our estimates of the performance of the CQ heuristic are optimistic.
**Complete bundling only heuristic**  Under the BO heuristic, the firm only offers the complete bundle denoted by $B = \{1, ..., n\}$, provided it is profitable. In this case, the only decision variable is the price of the complete bundle $r_B^{BO}$. The problem becomes:

$$
(P_{BO}) \quad \mathbb{E}\Pi(r_B^{BO}) = \max_{r_B^{BO}} \mathbb{E}\Pi(r_B^{BO}) \quad \text{such that } P_B \leq u
$$

where $u = \min_{i=1, ..., n} u_i$ and $\mathbb{E}\Pi(r_B^{BO}) = \left(1 - \frac{r_B^{BO}}{Q_B}\right)^b (r_B^{BO} - c_B)$. Lemma 8 gives the optimal solution to this problem.

**Lemma 8.** The optimal complete bundle price $r_B^*$ is equal to

$$
r_B^{*BO} = \begin{cases} 
\frac{Q_B + bC_B}{b+1} & \text{if } \frac{Q_B + bC_B}{(b+1)Q_B} < 1, \left(1 - \frac{Q_B + bC_B}{b+1}\right)^b \leq u \text{ and } Q_B > C_B \\
(1 - \frac{u}{b})Q_B & \text{if } \frac{Q_B + bC_B}{(b+1)Q_B} < 1, \left(1 - \frac{Q_B + bC_B}{b+1}\right)^b \leq u \text{ and } (1 - \frac{u}{b})Q_B > C_B \\
+\infty & \text{otherwise}
\end{cases}
$$

**Comparison of the heuristics**  We compare the heuristics on two dimensions. The first dimension is the how easy it is to calculate the prices and the second dimension is how easy it would be to implement these prices in practice. When the capacity is abundant, it is easy to calculate the prices from optimal solution and the IB, CQ and BO heuristics using an algorithm or a simple formula. In contrast, when capacity is limited, calculating the prices for the optimal policy, the IB and CO heuristics is complicated since it requires solving the constrained problem using KKT conditions and the number of constraints increases exponentially with $n$ and thus it quickly becomes intractable. The IB, CO and BO heuristics are easy to implement because the firm has to set only a small number of prices (at most $n$ for IB and CO and only one for BO). In contrast, implementing the optimal policy and the CQ heuristic may mean listing up to $2^n - 1$ different prices. Table 7 summarizes the comparison between the heuristics and the optimal policy.

<table>
<thead>
<tr>
<th>Ease of calculation</th>
<th>Optimal</th>
<th>IB</th>
<th>CO</th>
<th>CQ</th>
<th>BO</th>
</tr>
</thead>
<tbody>
<tr>
<td>abundant cap.</td>
<td>Easy</td>
<td>Easy</td>
<td>Hard</td>
<td>Easy</td>
<td>Easy</td>
</tr>
<tr>
<td>limited cap.</td>
<td>Hard</td>
<td>Hard</td>
<td>Hard</td>
<td>Easy</td>
<td>Easy</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ease of implementation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hard</td>
</tr>
<tr>
<td>Easy</td>
</tr>
<tr>
<td>Easy</td>
</tr>
<tr>
<td>Hard</td>
</tr>
<tr>
<td>Easy</td>
</tr>
</tbody>
</table>

**Table 7:** Comparison of the heuristics.
5.1 Performance of the heuristic strategies

When the retailer prices the products according to the heuristic policies, consumers may decide to buy more than one product. In particular, if the retailer prices only the components (as in the IB and CO heuristics), it is possible that some customers buy multiple components, hereby “building their own bundle”. For example, suppose \( n = 2 \) and assume that some consumers purchase both components 1 and 2. In this case we say that set \( S = \{1, 2\} \) receives a positive purchase probability so that the effective assortment is \( A = \{\{1, 2\}\} \). We say that the chosen component prices result in an effective pure bundling strategy. In other words, even though the firm sets prices for individual components only, the component prices can lead to an effective pure or mixed bundling strategy. An important question when analyzing the performance of the heuristics is whether or not their effective bundling strategy matches the optimal bundling strategy. When it does, we say the heuristic is able to “mimic” the optimal bundling strategy. However, the prices set by the heuristic may not be optimal even if it is able to mimic the optimal bundling strategy, as shown in the following example.

**Example 3.** Let \( n = 2, q_1 = 3.89, q_2 = 2.31, c_1 = 1.58, c_2 = 0.92, \alpha = 0.82, u_1 = u_2 = 1 \). Assume \( F \) is uniform on \([0, 1]\), i.e., \( b = 1 \).

<table>
<thead>
<tr>
<th></th>
<th>Optimal</th>
<th>IB</th>
<th>CO</th>
<th>CQ</th>
<th>BO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected profit</td>
<td>0.2246</td>
<td>0.1809</td>
<td>0.2009</td>
<td>0.2246</td>
<td>0.2161</td>
</tr>
<tr>
<td>Optimality gap</td>
<td>MB12</td>
<td>PC12</td>
<td>MB12</td>
<td>MB12</td>
<td>PB</td>
</tr>
<tr>
<td>Effective strategy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prices</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r_1 )</td>
<td>2.31</td>
<td>2.31</td>
<td>2.15</td>
<td>2.31</td>
<td>-</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>1.45</td>
<td>1.45</td>
<td>1.29</td>
<td>1.45</td>
<td>-</td>
</tr>
<tr>
<td>( r_B )</td>
<td>3.48</td>
<td>-</td>
<td>-</td>
<td>3.48</td>
<td>3.48</td>
</tr>
<tr>
<td>Purchase probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_1 )</td>
<td>1.29%</td>
<td>18.85%</td>
<td>9.64%</td>
<td>1.29%</td>
<td>0%</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>8.00%</td>
<td>8.00%</td>
<td>16.35%</td>
<td>8.00%</td>
<td>0%</td>
</tr>
<tr>
<td>( P_B )</td>
<td>17.56%</td>
<td>0%</td>
<td>9.21%</td>
<td>17.56%</td>
<td>22.00%</td>
</tr>
</tbody>
</table>

**Table 8:** Comparison of the heuristics

The IB and CO heuristics both only set prices for components 1 and 2 but the prices set by the CO heuristic are such that some consumers decide to create their own bundle, leading to an effective MB12 strategy. In other words, the CO heuristic is able to mimic the optimal bundling strategy. Yet the CO and IB heuristics are not optimal because the prices they set differ from the optimal prices. In contrast, the CQ heuristic is optimal as it sets exactly the same prices as the optimal policy.
The next two results establish the optimality of the CO and CQ heuristics in some specific cases.

**Proposition 9.** When $\alpha \leq 1$ and capacity is abundant, the CQ heuristic is optimal, that is, it is possible to obtain the optimal expected profit by pricing all products using equation (8).

However, in practice the solution provided by the CQ heuristic may not be easily implementable by the firm since it implies setting a price for up to $2^n - 1$ different products. Therefore it is important to study how well the other heuristics perform in that case.

**Proposition 10.** When $\alpha > 1$, the CO heuristic is optimal, that is, it is possible to obtain the optimal expected profit by pricing only the components, when (i) $n = 2$ and (ii) $n > 2$ and capacity is abundant.

When the quality relationship is super-additive, it is always possible to mimic the optimal effective assortment by pricing only the components. For example, suppose that the optimal strategy is MB1, meaning that, given the optimal prices, consumers buy only component 1 and the bundle $\{1,2\}$. Proposition 10 states that the prices of the two components could be set such that some consumers purchase component 1 only and some consumers “build their own bundle” by buying both components but no consumer buys component 2 only. In contrast, when the quality relationship is sub-additive, a full spectrum mixed bundling strategy (MB12) may be optimal. In that case, the retailer is unable to mimic the optimal bundling strategy because it would require three prices but there are only two individual components to price.

### 6 Numerical study

The purpose of this numerical is twofold. First we evaluate the value of bundling, i.e., how much extra profit the retailer can achieve by considering the bundling behavior of consumers for vertically differentiated goods. Second, we study the performance of the heuristic pricing policies presented in Section 5 and provide recommendations as to when they perform well.

We considered the following values: $n \in \{2,3,...,10\}$. For $n = 2$, we used $q_1 = 5$, $q_2 \in \{3,3.5,4,4.5\}$, $c_1 \in \{1,2,3,4,5,7.5,10.15\}$, $c_2 = c_1/\beta$ where $\beta \in \{0.5,1,...,5\}$, $u_1, u_2 \in \{0.25,0.5,0.75,1\}$, $\alpha \in \{0.5,0.6,...,1.2\}$, and $b \in \{0.25,0.5,...,1.75\}$. For $n = \{3,...,10\}$, we used $q_i = i$, $u_i = 1$, $c_i = kq_i^y$.
for \( i = 1, \ldots, n \) and \( k \in \{0.1, 0.2, \ldots, 0.9\}, \ y \in \{0.1, 0.2, \ldots, 3\}, \ \alpha \in \{0.1, 0.2, \ldots, 3\}, \) and \( b \in \{0.1, 0.2, \ldots, 3\}. \)

Overall we considered 2,230,720 problem instances. For each heuristic, we calculate optimality gaps with respect to the optimal strategy as follows: \( OG^h = \frac{\mathbb{E}\Pi^* - \mathbb{E}\Pi^h}{\mathbb{E}\Pi^*} \) where \( \mathbb{E}\Pi^* \) is the optimal expected profit and \( \mathbb{E}\Pi^h \) is the maximum expected profit values for heuristic \( h \in \{IB, CO, CQ, BO\}. \)

We first analyze the results for the two-component problems. The average optimality gaps for \( n = 2 \) are reported in Table 9.

<table>
<thead>
<tr>
<th>IB</th>
<th>CO</th>
<th>CQ</th>
<th>BO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha \leq 1 )</td>
<td>( \alpha &gt; 1 )</td>
<td>( \alpha \leq 1 )</td>
<td>( \alpha &gt; 1 )</td>
</tr>
<tr>
<td>abundant</td>
<td>3.50%</td>
<td>16.03%</td>
<td>0.49%</td>
</tr>
<tr>
<td>limited</td>
<td>3.95%</td>
<td>19.72%</td>
<td>0.73%</td>
</tr>
<tr>
<td>Average</td>
<td>7.82%</td>
<td>0.53%</td>
<td>6.20%</td>
</tr>
</tbody>
</table>

Table 9: Average optimality gaps of heuristic pricing strategies when \( n = 2 \).

By Proposition 10, we know that the CO heuristic is optimal when \( \alpha > 1 \). From Table 9, we see that it leads to very low optimality gaps (0.49% and 0.73% respectively under abundant and limited capacity) even when \( \alpha \leq 1 \). This suggests that the retailer can do very well by optimally pricing only the components. However these prices need to be carefully calculated, in a way that accounts for the fact that consumers may be willing to buy multiple products. If the retailer ignores this fact, a large portion of profit is left on the table, as evidenced by the high optimality gaps of the IB heuristic (7.82% on average). Using a simple pricing formula for pricing products can also lead to a large optimality gap since the CQ heuristic has an average percentage profit loss of 6.20% (which is a conservative estimate). Finally offering only the complete bundle is generally not a good strategy, even when consumers value the joint consumption of the components; the average optimality gap of the BO heuristic is still 22.67% and 24.16% when \( \alpha > 1 \), respectively under abundant and limited capacity.

Table 10 provides a breakdown of when each heuristic achieves the optimal profit in a two-component problem. The first column lists the optimal bundling strategy. The second and seventh columns (labeled “freq.” for \( \alpha \leq 1 \) and \( \alpha > 1 \) respectively) report the frequency with which these strategies are optimal for all problem instances with \( \alpha \leq 1 \) and \( \alpha > 1 \) respectively. The remaining columns report the percentage of time each heuristic achieves the optimal expected profit.

From Table 10 we see that the CO heuristic is always suboptimal when the optimal strategy is MB12 (which by Proposition 1 is only possible when \( \alpha \leq 1 \)), because it has only two decision variables
When \( \alpha \leq 1 \) and the optimal strategy is MB1 or MB2, the CO strategy may or may not be optimal depending on whether or not the optimal prices can be mimicked. In every other case, the CO strategy achieves the optimal expected profit. We also see that the BO strategy is only optimal when the optimal effective strategy is to offer the complete bundle or offer nothing at all but suboptimal in every other case. This is because the BO strategy sets only one price \( (r_{\{1\}, 2}) \) but the other six possible bundling strategies require at least two prices. The IB heuristic is only guaranteed to be optimal when the optimal strategy is to offer nothing, only component 1 (PC1) or only component 2 (PC2). For problems with limited capacity, the CQ heuristic never has a guarantee of optimality, except when the optimal assortment is the empty set.

Next we analyze the results for larger problems. The average optimality gaps for \( n > 2 \) are reported in Table 11. From Proposition 9 we know that the CO heuristic is optimal when \( \alpha > 1 \) and capacity is abundant. When capacity is limited, we are not able to calculate the prices from the CO heuristic in all problem instances and thus we did not report average optimality gaps. Another important observation from Table 11 is that the optimality gap of the IB heuristic, i.e., the value of bundling, grows quickly with the number of components in the product category. Moreover, the value remains very high (up to 9.48% when \( n = 10 \)) even when the quality relationship is sub-additive, which suggests that retailers can greatly benefit from considering bundling strategies even when consumers do not value the joint consumption of components more than the sum of individual consumptions. This demonstrates that bundling vertically differentiated products is a very promising (and currently untapped) opportunity for greater profits in retail. Finally we see from Table 11 that offering only the complete bundle leads

<table>
<thead>
<tr>
<th>Opt bund. strategy</th>
<th>freq.</th>
<th>IB</th>
<th>CO</th>
<th>CQ</th>
<th>BO</th>
<th>freq.</th>
<th>IB</th>
<th>CO</th>
<th>CQ</th>
<th>BO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nothing</td>
<td>35.16%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>15.94%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>PB</td>
<td>0.25%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>PC12</td>
<td>0.49%</td>
<td>36.84%</td>
<td>100%</td>
<td>31.58%</td>
<td>0%</td>
<td>3.59%</td>
<td>100%</td>
<td>100%</td>
<td>73.91%</td>
<td>0%</td>
</tr>
<tr>
<td>PC1</td>
<td>5.29%</td>
<td>100%</td>
<td>100%</td>
<td>93.84%</td>
<td>0%</td>
<td>16.72%</td>
<td>100%</td>
<td>100%</td>
<td>77.34%</td>
<td>0%</td>
</tr>
<tr>
<td>PC2</td>
<td>36.56%</td>
<td>100%</td>
<td>100%</td>
<td>89.49%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>MB12</td>
<td>2.02%</td>
<td>0%</td>
<td>0%</td>
<td>9.00%</td>
<td>0%</td>
<td>10.78%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>MB1</td>
<td>3.62%</td>
<td>0%</td>
<td>62.41%</td>
<td>56.83%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>MB2</td>
<td>16.60%</td>
<td>7.76%</td>
<td>56.35%</td>
<td>82.33%</td>
<td>0%</td>
<td>42.79%</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Avg</td>
<td>78.48%</td>
<td>89.37%</td>
<td>88.90%</td>
<td>35.41%</td>
<td></td>
<td>36.25%</td>
<td>100%</td>
<td>31.52%</td>
<td>26.11%</td>
<td></td>
</tr>
</tbody>
</table>
to a large optimality gap even when $\alpha > 1$ so it is generally not a good idea to restrict the choice of the consumers and force them to buy all the components or nothing.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha \leq 1$</th>
<th>$\alpha &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IB  CO  CQ  BO</td>
<td>IB  CO  CQ  BO</td>
</tr>
<tr>
<td>3</td>
<td>4.61% (* ) 0% 78.68%</td>
<td>71.31% 0% 52.26% 7.58%</td>
</tr>
<tr>
<td>4</td>
<td>5.95% (* ) 0% 82.05%</td>
<td>77.61% 0% 60.54% 8.56%</td>
</tr>
<tr>
<td>5</td>
<td>6.95% (* ) 0% 83.78%</td>
<td>81.13% 0% 65.69% 9.22%</td>
</tr>
<tr>
<td>6</td>
<td>7.71% (* ) 0% 84.82%</td>
<td>83.33% 0% 69.22% 9.66%</td>
</tr>
<tr>
<td>7</td>
<td>8.30% (* ) 0% 85.53%</td>
<td>84.81% 0% 71.79% 10.00%</td>
</tr>
<tr>
<td>8</td>
<td>8.78% (* ) 0% 86.04%</td>
<td>85.87% 0% 73.75% 10.29%</td>
</tr>
<tr>
<td>9</td>
<td>9.17% (* ) 0% 86.42%</td>
<td>86.64% 0% 75.28% 10.51%</td>
</tr>
<tr>
<td>10</td>
<td>9.48% (* ) 0% 86.70%</td>
<td>87.22% 0% 76.52% 10.67%</td>
</tr>
</tbody>
</table>

(*) CO prices not known in all cases

Table 11: Average optimality gaps of heuristic pricing strategies when $n > 2$ and capacity is abundant.

Next, we study how the value of bundling, i.e., the optimality gap of the IB heuristic, varies with $\alpha$ and $b$ around the following base case: , $n = 2$, $q_1 = 5$, $q_2 = 4$, $c_1 = 1 = c_2$, $\alpha = 0.7$, $u_1 = u_2 = 1$ and $b = 1$ (the behavior was similar for other problem instances).

On Figure 10 we see that, as expected, the value of bundling increases with $\alpha$: the more consumers value the joint consumption of the components, the more important it is for the firm to offer bundling options. Figure 11 shows that the value of bundling also increase as more and more consumers value the quality attribute (which corresponds to a decrease in $b$). Hence, the benefits of bundling increase over the life cycle of a product.

In summary, our numerical study suggests that a retailer can greatly improve profits by bundling vertically differentiated products. This is true even in the case where consumers get less utility from
the joint consumption of the components than the sum of utilities from individual consumption (sub-additive quality relationship). However it is important to price the products which are offered optimally.

When the quality relationship is sub-additive, a simple pricing formula (i.e., the CQ heuristic) is optimal but difficult to implement for large product categories (because up to $2^n - 1$ prices have to be listed).

When the quality relationship is super-additive, the retailer can achieve the optimal profit by carefully pricing the components, that is, by taking the consumers’ bundling behavior into consideration and having the consumers build their own bundles.

7 Conclusion

In this paper, we characterize the optimal bundling strategies in settings with two or more vertically differentiated products, a non-uniform distribution of customer valuation, positive variable costs, with and without capacity constraints. We provide closed form expressions for the optimal prices and assortments, which is noteworthy since most papers on bundling either do not consider the mixed bundling strategy or rely on numerical study to get the optimal prices. Our analysis also provides several insightful results. We show that, when components are in abundant supply, each bundling strategy (i.e., pure components, pure bundling or mixed bundling) could be optimal, which is in contrast to previous work showing that the pure bundling strategy is always optimal with zero variable costs. We find that the optimal bundling strategy does not depend on the customer valuation distribution but depends on the cost-quality ratios of the components and we prove that the optimal assortment does not include dominated products, but dominated components may be offered as part of a bundle. In contrast, when components are in limited supply, the bundling strategy does depend on the customer valuation distribution and the optimal assortment may include dominated products. Further, our results show that component scarcity may make bundling more or less attractive. Finally, we examine a number of heuristic pricing strategies. We show that, in some cases, the retailer can achieve high profits by pricing only the components. However these prices should be calculated carefully; ignoring the consumers’ bundling purchase behavior can lead to significant profit loss. This reveals that bundling vertically differentiated products is an untapped opportunity to improve profits for retailers. Consider the example of milk. Retailers traditionally divide their consumer base into more-price sensitive consumers who are willing to buy soon-to-expire milk at a discount and quality-driven consumers who prefer to pay
full price for freshness. By recognizing that the same consumer may buy both types of milk, one for immediate consumption (or for cooking purposes) and one for use later during the week, the retailer can better price the products, entice consumers to buy more and ultimately increase the total profit.

One limitation of our result is that we consider only vertically differentiated products. In practice, product categories may include both horizontal and vertical dimensions (e.g., MP3 players come in different combinations of colors and memory sizes). Studying how the two dimensions impact the optimal bundling and pricing strategies for a large product category is an interesting question since the research on bundling with both types of differentiation has arrived at different conclusions. Another limitation of our work is that we assume that consumers can derive utility from at most one unit of each component. Allowing consumers to buy multiple units of multiple components is an interesting extension for non-perishable products in the retail industry. We leave this direction for future research. Finally, we consider a monopolistic retailer. It would be interesting to examine how competition impacts the optimal bundling strategy.

References


Appendix A: Proofs

Proof of Lemma 1  The proof is by contradiction. Suppose we have $S_1, S_2 \in A^*$ such that $S_1 \cap S_2 = \emptyset$. The product numbering is such that $Q_{S_1} < Q_{S_2}$. Let $S = S_1 \cup S_2$. We have $\frac{r_{S}}{Q_{S}} \leq \frac{r_{S_1} + r_{S_2}}{Q_{S_1} + Q_{S_2}} < \frac{r_{S_1}}{Q_{S_1} + Q_{S_2}} < \frac{r_{S_2}}{Q_{S_2}}$, where the first inequality is by (2), the second inequality stems from $Q_S = (\sum_{j \in S_1} q_j + \sum_{j \in S_2} q_j) \alpha > (\sum_{j \in S_1} q_j) \alpha + (\sum_{j \in S_2} q_j) \alpha = Q_{S_1} + Q_{S_2}$ when $\alpha > 1$ and the third inequality is because, by (3), we have $\frac{r_{S_1}}{Q_{S_1}} < \frac{r_{S_2}}{Q_{S_2}}$ which implies that $\frac{r_{S_1} + r_{S_2}}{Q_{S_1} + Q_{S_2}} < \frac{r_{S_2}}{Q_{S_2}}$. But $\frac{r_{S}}{Q_{S}} \leq \frac{r_{S_2}}{Q_{S_2}}$ implies that every consumer who gets a positive utility from product $S_2$ gets an even bigger utility from product $S$ therefore no consumer will buy product $S_2$, which contradicts the fact that $S_2 \in A^*$.

Lemma 9. Let $\phi(x, y) = (x + y)^\alpha - x^\alpha$ where $\alpha, x, y > 0$. $\phi(x, y)$ is increasing in $y$ for all $\alpha > 0$. Further it is decreasing in $x$ for $0 < \alpha < 1$ and increasing in $x$ for $\alpha > 1$.

Proof. We have $\frac{\partial \phi(x, y)}{\partial y} = \alpha (x + y)^{\alpha - 1} > 0$ so $\phi$ is increasing in $y$. Further, $\frac{\partial \phi(x, y)}{\partial x} = \alpha [(x + y)^{\alpha - 1} - x^{\alpha - 1}]$ which is less than 0 for $0 < \alpha < 1$ and greater than 0 for $\alpha > 1$ since $z^{\alpha - 1}$ is decreasing in $z$ for $0 < \alpha < 1$ and increasing in $z$ for $\alpha > 1$.

Proof of Proposition 2

When components are available in infinite supply, problem $(P)$ only has constraints (2). If we relax these constraints, the problem is equivalent to that considered in Pan & Honhon (2012), i.e., a firm needs to choose the optimal selling prices for $N$ vertically differentiated products. Let $\tilde{\vec{r}}^{PH}$ denote the set of optimal prices for the problem in Pan & Honhon (2012) and let $A^{PH} = A(\tilde{\vec{r}}^{PH})$ be the corresponding optimal effective assortment. By Lemma 6 in Pan & Honhon (2012), the price vector $\tilde{\vec{r}}^{PH}$ satisfies (8) and by Lemma 2 in Pan & Honhon (2012), the effective assortment $A^{PH}$ satisfies (4)-(7). In this proof, we show that $\tilde{\vec{r}}^{PH}$ satisfies constraints (2) so that $\vec{r}^* = \tilde{\vec{r}}^{PH}$ and $A^* = A^{PH}$.

Second, we show the products in $A^{PH}$ are nested when $\alpha > 1$. We do so by contradiction: assume there exists $S, S' \in A^{PH}$ such that $Q_S < Q_{S'}$ but $S \notin S'$. Let $T = S \cap S'$. By Lemma 1 we know that $T \neq \emptyset$. By Lemma 9, we have $Q_{S \cup S'} - Q_{S'} = \phi \left( \sum_{i \in S'} q_i, \sum_{i \in S \setminus S'} q_i \right) > \phi \left( \sum_{i \in T} q_i, \sum_{i \in S \setminus S'} q_i \right) = Q_S - Q_T$ since $\phi$ is increasing in its first argument when $\alpha > 1$.  

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It is easy to see that (4) implies that
\[
\frac{C_{s_i}}{Q_{s_i}} < \frac{C_{s_j} - C_{s_k}}{Q_{s_j} - Q_{s_k}} \quad \text{for } i < k \leq m
\] (12)

Since \( S, S' \in A^{PH} \), this implies that (i) \( \frac{C_S}{Q_S} < \frac{C_{s'} - C_S}{Q_{s'} - Q_S} \). Next we show that (ii) \( \frac{C_{s'} - C_S}{Q_{s'} - Q_S} < \frac{C_{s''} - C_{s'}}{Q_{s''} - Q_{s'}} \) by contradiction. If this was not true, then \( S' \) would not belong to \( A^* \) as it would satisfy (6).

Conditions (i) and (ii) imply that
\[
\frac{C_{s'}}{Q_{s'}} = \frac{C_S}{Q_S} = \frac{C_{s'} - C_S}{Q_{s'} - Q_S} < \frac{\sum_{i \in S \setminus S'} c_i}{Q_{s'} - Q_S} < \frac{\sum_{i \in S' \setminus S} c_i}{Q_{s'} - Q_S} = \frac{C_S - C_{s'}}{Q_S - Q_{s'}}
\] (13)

If \( T \in A^* \), we must have \( \frac{C_{s'} - C_T}{Q_{s'} - Q_T} < \frac{C_S - C_T}{Q_S - Q_T} \) by (4), which contradicts (13). If \( T \notin A^* \), we must have \( \frac{C_T}{Q_T} > \frac{C_S}{Q_S} \) as implied (6), but \( \frac{C_{s'}}{Q_{s'}} < \frac{C_S - C_{s'}}{Q_S - Q_{s'}} \) implies that \( \frac{C_T}{Q_T} < \frac{C_S}{Q_S} \), which is a contradiction. This proves that the products in \( A^{PH} \) are nested.

Now, we prove that \( \tau^{PH} \) satisfy constraints (2). When \( \alpha > 1 \), at least one price on the right hand side of (2) must be infinite since the products in \( A^{PH} \) are nested, hence the constraints are always satisfied. When \( \alpha \leq 1 \), consider sets \( S_j \) and \( S_k \) such that \( S_j \cup S_k = S \). If \( S_j \cap S_k \neq \emptyset \), we have \( r_{s_j} + r_{s_k} \geq r_{s_j \setminus s_k} + r_{s_k} \) because of (5), we have that \( r_{s_j} \leq r_{s_j} \) for \( S_i \subset S_j \). Therefore we only need to prove that constraints (2) are satisfied for \( S_j \) and \( S_k \) such that \( S_j \cap S_k = \emptyset \). We have:
\[
r_S = \frac{bC_S + Q_S}{1 + b} \leq \sum_{j=1}^{k} \frac{bC_{s_j} + Q_{s_j}}{1 + b} = \sum_{j=1}^{k} r_{s_j}
\] (14)

where the inequality follows from the fact that \( Q_S = (\sum_{i \in S} q_i)^\alpha = (\sum_{j=1}^{k} \sum_{i \in S_j} q_i)^\alpha \leq \sum_{j=1}^{k} \left( \sum_{i \in S_j} q_i \right)^\alpha = \sum_{j=1}^{k} Q_{s_j} \) when \( \alpha \leq 1 \). This proves that \( \tau^{PH} \) satisfies constraints (2) and therefore \( \tau^* = \tau^{PH} \) and \( A^* = A^{PH} \), which completes the proof of (a). Also since the products in \( A^{PH} \) are nested, this is also true for the products in \( A^* \), proving (b).

**Proof of Lemma 2** From (14) with strict inequalities we can prove that \( r_S < r_{s_j} + r_{s_k} \) for all \( S \subseteq \{1, \ldots, n\} \) and \( S_j \cup S_k = S, j \neq k \), when \( \alpha < 1 \). The result can be obtained by using the same inequality for \( r_{s_j} \) and \( r_{s_k} \).
Proof of Proposition 3

The result follows almost directly from Lemma 4 from Pan & Honhon (2012). The only differentiating element is that our definition of dominated products allows for the case where the dominating product has the same cost as the dominated product. But it is straightforward to see that the proof from Pan & Honhon (2012) can be extended to encompass that case as well.

Proof of Lemma 3

Suppose \( A^* = \{ S_{j_1}, ..., S_{j_m} \} \), where \( j_1 < j_2 < ... < j_m \). We know from Proposition 2 that the optimal effective assortment does not depend on \( F \). From (8), it is easy to see that optimal prices are decreasing in \( b \). The total market share of the product category is equal to \( \sum_{i=1}^{m} P_{S_{j_i}} = 1 - F \left( \frac{r_{S_{j_1}}}{Q_{S_{j_1}}} \right) = \left( 1 - \frac{r_{S_{j_1}}}{Q_{S_{j_1}}} \right)^b = \left( \frac{b}{b+1} \right)^b \left( \frac{Q_{S_{j_1}} - C_{S_{j_1}}}{Q_{S_{j_1}}} \right)^b \), which is increasing in \( b \) for all \( j_1 \).

Proof of Proposition 4

When all the components have the same cost, only product \( T_i \) for \( i \in \{ 1, ..., n \} \) is non-dominated and thus only those products could be included in the optimal assortment from Proposition 3.

Proof of Lemma 4

Barghava & Choudhary (2001) show (Lemma 1) that (9) is a sufficient condition. Next we show that (9) is a necessary condition, i.e., if (9) is not satisfied, then the optimal set \( S^* \) does not includes the bundle. We prove the result by contradiction: suppose that (9) is violated but \( A^* \) contains \( T_n \). Let \( A^* = \{ T_{j_1}, ..., T_{j_m} \} \) such that \( j_1 < j_2 < ... < j_m \) and \( j_m = n \). First suppose that (9) is not satisfied because \( \frac{nc}{Q_{T_n}} = \frac{C_{T_n}}{Q_{T_n}} > 1 \). In this case, (4) is violated therefore \( S^* \) cannot be optimal. Now suppose that there exists \( k \in \{ 1, ..., n-1 \} \) such that \( \frac{c(n-k)}{Q_{T_n} - Q_{T_k}} = \frac{C_{T_n} - C_{T_k}}{Q_{T_n} - Q_{T_k}} > 1 \). If \( k \in S^* \) and \( k = j_m - 1 \) then (4) is violated, which is a contradiction. If \( k \in S^* \) and \( k < j_m - 1 \), we have \( 1 < \frac{C_{T_n} - C_{T_k}}{Q_{T_n} - Q_{T_k}} < \frac{C_{T_n} - C_{T_{j_m-1}}}{Q_{T_n} - Q_{T_{j_m-1}}} \) therefore once again (4) is violated. If \( k \notin S^* \), there are two subcases: if \( m = 1 \) such that \( A^* = \{ T_n \} \), then (6) implies that \( \frac{C_{T_k}}{Q_{T_k}} > \frac{C_{T_n} - C_{T_k}}{Q_{T_n} - Q_{T_k}} > 1 \) but this would imply that \( \frac{C_{T_n}}{Q_{T_n}} > 1 \) which is a contradiction to (4). If \( m > 1 \), (6) implies that \( \frac{C_{T_n} - C_{T_{j_m-1}}}{Q_{T_n} - Q_{T_{j_m-1}}} > \frac{C_{T_n} - C_{T_k}}{Q_{T_n} - Q_{T_k}} > 1 \) which again contradicts (4).

Proof of Lemma 5

When \( c = 0 \), (9) is satisfied, therefore, the optimal set includes the bundle based on Lemma 4. If any other product \( S \) was also included in \( A^* \), it would contradict condition (4). Therefore, \( A^* = \{ T_n \} \).

Proof of Proposition 5
In order to prove Proposition 5 we first state a lemma.

Lemma 10. There exist two threshold values $\alpha$ and $\overline{\alpha}$ such that $\Delta Q_{T_1} > \Delta Q_{T_2} > \ldots > \Delta Q_{T_n}$ for $\alpha < \alpha$ and $\Delta Q_{T_1} < \Delta Q_{T_2} < \ldots < \Delta Q_{T_n}$ for $\alpha > \overline{\alpha}$. Further, we have $1 \leq \alpha \leq \overline{\alpha}$.

Proof. We have $\Delta Q_{T_j} = Q_{T_j} - Q_{T_{j-1}} = (Q_{T_{j-1}} + q_j)\alpha - (Q_{T_{j-1}})\alpha = \phi(Q_{T_j}, q_j)$.

First, we prove that $\Delta Q_{T_1} > \Delta Q_{T_2} > \ldots > \Delta Q_{T_n}$ for $0 < \alpha < 1$. In this case, we have $\Delta Q_{T_j} = \phi(Q_{T_j}, q_j) > \phi(Q_{T_j}, q_{j+1}) > \phi(Q_{T_{j+1}}, q_{j+1}) = \Delta Q_{T_{j+1}}$. where the first inequality is because $\phi$ is increasing in its second argument (and $q_{j+1} < q_j$) and the second inequality is because $\phi$ is decreasing in its first argument (and $Q_{T_{j+1}} = \sum_{i=1}^{j+1} q_i > Q_{T_j} = \sum_{i=1}^{j} q_i$).

Second, we have:

$$\Delta Q_{T_{j+1}} - \Delta Q_{T_j} = \left(\sum_{i=1}^{j} q_i + q_{j+1}\right)\alpha - 2\left(\sum_{i=1}^{j} q_i\right)\alpha + \left(\sum_{i=1}^{j} q_i - q_j\right)\alpha > 0$$

$$\left(1 + \frac{q_{j+1}}{x}\right)\alpha + \left(1 - \frac{q_j}{x}\right)\alpha > 2$$

where $x = \sum_{i=1}^{j} q_i$. The limit of the left-hand side of (15) as $\alpha$ tends to $+\infty$ is $+\infty$ so there must exist a value of $\alpha$ so that (15) is satisfied for $j = 1, \ldots, n - 1$. \[\square\]

Equipped with Lemma 10, we can now provide the proof of Proposition 5.

Proof of (i): Consider $\alpha \in [0, \alpha]$. Suppose (contradiction) the optimal effective assortment is $A^* = \{T_1, T_2, \ldots, T_i, T_k, \ldots\}$, where $k > i$, that is products $T_{i+1}, \ldots, T_{k+1}$ are not included. From Lemma 10 we have $Q_{T_{i+1}} - Q_{T_i} > Q_{T_{i+2}} - Q_{T_{i+1}} > \ldots > Q_{T_k} - Q_{T_{k-1}}$. By adding up the terms after the first inequality we get, $(k-i-1)(Q_{T_{i+1}} - Q_{T_i}) > (Q_{T_{i+2}} - Q_{T_{i+1}}) + \ldots + (Q_{T_{k+1}} - Q_{T_k}) = Q_{T_k} - Q_{T_{i+1}}$. And therefore we have $rac{1}{Q_{T_{i+1}} - Q_{T_i}} < \frac{k-i-1}{Q_{T_k} - Q_{T_{i+1}}}$, which in turn implies $\frac{C_{T_{i+1}} - C_{T_i}}{Q_{T_{i+1}} - Q_{T_i}} \leq \frac{k-i-1}{Q_{T_k} - Q_{T_{i+1}}}$. This proves that $A^* = \{T_1, \ldots, T_i\}$ for some $j^*$. Next we show that $j^*$ must satisfy $c < \Delta Q_{T_{j^*}}$. If not, we would have $rac{c}{Q_{T_{j^*}} - Q_{T_{j^*-1}}} = \frac{C_{T_{j^*}} - C_{T_{j^*-1}}}{Q_{T_{j^*}} - Q_{T_{j^*-1}}} \geq 1$ which would contradict (6). Finally we prove that $j^*$ must be the largest integer satisfying condition $c < \Delta Q_{j^*}$. Suppose not, then it must be that $c < \Delta Q_{j^*+1}$, which implies that $\frac{C_{T_{j^*+1}} - C_{T_{j^*}}}{Q_{T_{j^*+1}} - Q_{T_{j^*}}} = \frac{c}{Q_{T_{j^*+1}} - Q_{T_{j^*}}} < 1$ but this contradicts (7).

Proof of (ii): Suppose that $\alpha > \overline{\alpha}$. We prove that $A^* = \{T_n\}$ satisfies the conditions of Proposition...
Since $nc < QT_n$, we have $\frac{CT_n}{QT_n} < 1$, therefore (4) and (5) hold. Now consider the sequence $QT_1, QT_2 - QT_1, ..., QT_n - QT_{n-1}$. By Lemma 10 this sequence is increasing. Therefore the average of the sum of the first $k$ elements, for $k = 1, ..., n$ in this sequence should be less or equal to the average of the $n$ elements, i.e., $\frac{Q_{Tk}}{n} < \frac{Q_{Tn}}{n}$. This implies that $\frac{kC}{Q_{Tk}} > \frac{(n-k)c}{Q_{Tn}-Q_{Tk}}$ and therefore we have $\frac{CT_n - CT_k}{Q_{Tn}-Q_{Tk}} = \frac{(n-k)c}{Q_{Tn}-Q_{Tk}}$ for $k = 1, ..., n - 1$ which is (6). All the conditions from Proposition 2 are met and therefore $A^*$ is optimal.

Proof of Lemma 6

Proof. From (8), we have $r^*_T = \frac{jc + Qr_j}{1 + b}$ which is increasing in $c$.

When $\alpha < \alpha$, by Proposition 5 we have $\Delta Q_{T_1} > ... > \Delta Q_{T_n}$ and $A^* = \{T_1, ..., T_j\}$ where $j^*$ is the largest value of $j$ which satisfies $c < \Delta Q_{T_j}$. As $c$ increases, this value of $j^*$ can only decrease therefore, fewer products are included in the optimal assortment. The total market share is $\sum_{i=1}^{j^*} P_{T_i} = 1 - F\left(\frac{r_{T_1}}{Q_{T_1}}\right) = \left(1 - \frac{r_{T_1}}{Q_{T_1}}\right)^b = \left(1 - \frac{Q_{T_1} - c}{Q_{T_1}}\right)^b$ which is decreasing in $c$.

When $\alpha > \alpha$, by Proposition 5 we have $A^* = \{T_n\}$. If $c$ increases past $Q_{T_n}/n$ then it becomes optimal not to offer any product so the optimal number of products also decreases in $c$. The total market share is $P_{T_n} = 1 - F\left(\frac{r_{T_n}}{Q_{T_n}}\right) = \left(1 - \frac{r_{T_n}}{Q_{T_n}}\right)^b = \left(1 - \frac{Q_{T_n} - cn}{Q_{T_n}}\right)^b$ which is decreasing in $c$.

Proof of Lemma 7

Proof. From (8), we have $r^*_T = \frac{jc + (q_1 + ... + q_j)\alpha}{1 + b}$ which is increasing in $\alpha$.

Proof of Proposition 8

Proof. First consider Case (1) with $\alpha \leq 1$. In this case we have $Q_{\{1,2\}} \leq Q_{\{1\}} + Q_{\{2\}}$ which implies that $\frac{Q_{\{1\}}}{Q_{\{2\}}} < \frac{Q_{\{1\}}}{Q_{\{2\}} - Q_{\{1\}}}$ and there are three sub-cases: (1.1) $\beta \leq \frac{Q_{\{1\}}}{Q_{\{2\}}}$, (1.2) $\frac{Q_{\{1\}}}{Q_{\{2\}}} \leq \beta \leq \frac{Q_{\{1\}} - Q_{\{2\}}}{Q_{\{1,2\}} - Q_{\{1\}}}$ and (1.3) $\beta > \frac{Q_{\{1\}} - Q_{\{2\}}}{Q_{\{1,2\}} - Q_{\{1\}}}$. In Case (1.1), we have $\frac{c_1}{Q_{\{1\}}} \leq \frac{c_1 + c_2}{Q_{\{1,2\}}} \leq \frac{c_2}{Q_{\{2\}}}$ and $\frac{c_1}{Q_{\{1\}}} \leq \frac{c_2}{Q_{\{1,2\}} - Q_{\{1\}}}$. If $\frac{c_2}{Q_{\{1,2\}} - Q_{\{1\}}} < 1$, the necessary and sufficient conditions for $A^* = \{\{1\}, \{2\}\}$ shown in Proposition 2 are satisfied. If $\frac{c_1}{Q_{\{1\}}} < 1 < \frac{c_2}{Q_{\{1,2\}} - Q_{\{1\}}}$, the necessary and sufficient conditions for $A^* = \{\{1\}\}$ shown in
Proposition 2 are satisfied. In Case (1.2), we have $\frac{c_2}{Q(2)} < \frac{c_1-c_2}{Q(1)-Q(2)} < \frac{c_2}{Q(1)}$. If $\frac{c_2}{Q(1)-Q(2)} < 1$, the necessary and sufficient conditions for $A^* = \{1, 2\}$ shown in Proposition 2 are satisfied. If $\frac{c_1-c_2}{Q(1)-Q(2)} < 1 < \frac{c_2}{Q(1)}$, the necessary and sufficient conditions for $A^* = \{1\}$ shown in Proposition 2 are satisfied. If $\frac{c_2}{Q(1)-Q(2)} < 1 < \frac{c_1-c_2}{Q(1)-Q(2)}$, the necessary and sufficient conditions for $A^* = \{1, 2\}$ shown in Proposition 2 are satisfied. In Case (1.3), we have $\frac{c_2}{Q(2)} < \frac{c_1}{Q(1)} < \frac{c_2}{Q(1)-Q(2)}$ and $\frac{c_1}{Q(1)} < 1$, the necessary and sufficient conditions for $A^* = \{1\}$ shown in Proposition 2 are satisfied. If $\frac{c_2}{Q(2)} < 1 < \frac{c_1}{Q(1)-Q(2)}$, the necessary and sufficient conditions for $A^* = \{2\}$ shown in Proposition 2 are satisfied.

Next consider Case (2) with $\alpha > 1$. In this case we have $Q_{1,2} > Q_{1} + Q_{2}$ which implies that $\frac{Q(1)}{Q(1)} < \frac{Q(1)]-Q(2)}{Q(2)}$ and there are three sub-cases: (2.1) $\beta \leq \frac{Q(1)}{Q(1)}$, (2.2) $\frac{Q(1)}{Q(1)} < \beta \leq \frac{Q(1)]-Q(2)}{Q(2)}$ and (2.3) $\beta > \frac{Q(1)]-Q(2)}{Q(2)}$. Case (2.1) is identical to case (1.1). In Case (2.2), we have $\frac{c_1}{Q(1)} > \frac{c_2}{Q(2)}$ and $\frac{c_2}{Q(2)} > \frac{c_1}{Q(1)-Q(2)}$. If $\frac{c_1+c_2}{Q(1)} < 1$, then the necessary and sufficient conditions for $A^* = \{1, 2\}$ shown in Proposition 2 are satisfied. Case (2.3) is identical to case (1.3).

\[\square\]

Proof of Proposition 7 The results directly follow from Proposition 5.

Proof of Proposition 9

For brevity, we sketch the proof here. We calculate the algebraic conditions for each possible combination of binding constraints. For example, if capacity constraints are non-binding, then the corresponding bundling strategy is MB12 and the solution does not use all the available capacity. By simplifying the KTT conditions we can write out the mathematical conditions for this case. We repeat the process for every case then compare the conditions to see when they are mutually exclusive. For example, we find that an MB12 strategy is only possible when $\alpha \leq 1$. On the other hand, a PB strategy is only possible when $\alpha > 1$. Therefore we conclude that it is never possible to switch from an MB12 to a PB strategy as the supply of component switches from being abundant to scarce. The detailed proof, which includes all the conditions, is available from the authors upon request.

\[\square\]

Proof of Proposition 10 The result follows directly from Proposition 2.
By Proposition 2 the products in $A^*$ are nested. To simplify the notation let us assume that $A^* = \{\{1, ..., j_1\}, \{1, ..., j_2\}, ..., \{1, ..., j_m\}\}$ (this is without loss of generalization). The optimal prices $\vec{r}^*$ are given by (8). The proof is by construction: we suggest a way to calculate component prices $r_{CO}^1, ..., r_{CO}^m$ then show that these prices achieve the optimal expected profit. Let $\theta_1 = \frac{r_{\{1, ..., j_1\}}^*}{Q_{\{1, ..., j_1\}}}$ and $\theta_k = \frac{r_{\{1, ..., j_{k-1}\}}^* - r_{\{1, ..., j_{k-1}\}}^*}{Q_{\{1, ..., j_k\}} - Q_{\{1, ..., j_{k-1}\}}}$ for $k = 2, ..., m$. We construct the component prices as follows:

$$r_{CO}^k = \begin{cases} 
\theta_1 \left( Q_{\{1, ..., k\}} - Q_{\{1, ..., k-1\}} \right) & k = 1, ..., j_1 \\
\theta_2 \left( Q_{\{1, ..., k\}} - Q_{\{1, ..., k-1\}} \right) & k = j_1 + 1, ..., j_2 \\
... & \\
\theta_m \left( Q_{\{1, ..., k\}} - Q_{\{1, ..., k-1\}} \right) & k = j_m - 1 + 1, ..., j_m \\
+\infty & k > j_m 
\end{cases}$$

(16)

These prices are such that $r_{CO}^1 + ... + r_{CO}^k = r_{\{1, ..., j_k\}}^*$ for $k = 1, ..., m$. Also $\frac{r_{CO}^1 + ... + r_{CO}^k}{Q_{\{1, ..., k\}}} = \theta_1$ for $k = 1, ..., j_1$, $\frac{r_{CO}^1 + ... + r_{CO}^k}{Q_{\{1, ..., k\}} - Q_{\{1, ..., j_1\}}} = \theta_2$ for $k = j_1 + 1, ..., j_2$, ... and $\frac{r_{CO}^1 + ... + r_{CO}^k}{Q_{\{1, ..., k\}} - Q_{\{1, ..., j_m\}}} = \theta_m$ for $k = j_m - 1 + 1, ..., j_m$.

To prove that these component prices achieve the optimal expected profit we only need to prove that any $S \not\subset A^*$ has $P_S = 0$ given that its selling price is $\sum_{i \in S} r_{CO}^i$. First, if $S$ contains a component $k > j_m$, then its price is infinite and therefore its purchase probability is zero. Otherwise find the value of $k \in \{1, ..., m\}$ such that $Q_{\{1, ..., j_{k-1}\}} < Q_S < Q_{\{1, ..., j_k\}}$. We prove that $Q_S \theta_k - r_S \leq Q_{\{1, ..., j_k\}} \theta_k - r_{\{1, ..., j_k\}}^*$, which is equivalent to $r_S - r_{\{1, ..., j_k\}}^* \geq (Q_S - Q_{\{1, ..., j_k\}}) \theta_k$. This implies that $P_S = 0$ because every customer with $\theta > \theta_k$ gets a higher utility from $\{1, ..., k\}$ than $S$ and every customer with $\theta < \theta_k$ gets a higher utility from $\{1, ..., k - 1\}$ than $S$. Let $S_1 = \{1, ..., j_1\} \cap S$ and $S_k = \{j_{k-1} + 1, ..., j_k\} \cap S$ for
\( k = 2, \ldots, m \). We have \( S = \bigcup_{k=1}^{m} S_k \).

\[
\begin{align*}
r_S - r_{\{1, \ldots, j_k\}}^* &= \sum_{i \in S} r_i^{CO} - \sum_{i=1}^{j_k} r_i^{CO} \\
&= \sum_{l=k+1}^{m} \sum_{i \in S_l} r_i^{CO} - \sum_{l=1}^{k} \sum_{i \in \{j_{l-1} + 1, \ldots, j_l\} \setminus S} r_i^{CO} \\
&\geq \theta_k \left( \sum_{l=k+1}^{m} \sum_{i \in S_l} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) - \sum_{l=1}^{k} \sum_{i \in \{j_{l-1} + 1, \ldots, j_l\} \setminus S} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) \right)
\end{align*}
\]

where the last inequality follows by \( \theta_l \geq \theta_k \) for \( l = k+1, \ldots, m \) and \( \theta_l \leq \theta_k \) for \( l = 1, \ldots, k \). So we need to prove that:

\[
\sum_{l=k+1}^{m} \sum_{i \in S_l} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) - \sum_{l=1}^{k} \sum_{i \in \{j_{l-1} + 1, \ldots, j_l\} \setminus S} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) \geq Q_S - Q_{\{1, \ldots, j_k\}}
\]

\[
\Leftrightarrow \sum_{l=k+1}^{m} \sum_{i \in S_l} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) + \sum_{l=1}^{k} \sum_{i \in S_l} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) \geq Q_S
\]

\[
\Leftrightarrow \sum_{i \in S} (Q_{\{1, \ldots, i\}} - Q_{\{1, \ldots, i-1\}}) \geq Q_S
\]

which is true when \( \alpha > 1 \).

\( \square \)

Appendix B: Tables
<table>
<thead>
<tr>
<th>Bundling strategy</th>
<th>Component. const. binding</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_{12} )</th>
<th>Necessary condition(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PB</td>
<td>none</td>
<td>-</td>
<td>-</td>
<td>( Q_{12} + c_{12} )</td>
<td>( \alpha &gt; 1 )</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>( Q_{12}(1 - u_1) )</td>
<td>( \alpha &gt; 1 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>( Q_{12}(1 - u_2) )</td>
<td>( \alpha &gt; 1 )</td>
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<tr>
<td></td>
<td>1 &amp; 2</td>
<td>-</td>
<td>-</td>
<td>( Q_{12}(1 - u_1) )</td>
<td>( \alpha &gt; 1, u_1 = u_2 )</td>
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<tr>
<td>PC1</td>
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<td>-</td>
<td>( Q_{12} + c_{12} )</td>
<td>( \alpha \leq 1 )</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>( r_2 + (Q_1 - Q_2)(1 - u_1) )</td>
<td>( Q_{2} + c_{2} )</td>
<td>-</td>
<td>( \alpha \leq 1 )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Q_{2} + c_{2} )</td>
<td>( \frac{Q_2}{Q_1} (r_1 - u_2(Q_1 - Q_2)) )</td>
<td>-</td>
<td>( \alpha \leq 1 )</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>( r_2 + (Q_1 - Q_2)(1 - u_1) )</td>
<td>( Q_2(1 - (u_1 + u_2)) )</td>
<td>-</td>
<td>( \alpha \leq 1 )</td>
</tr>
<tr>
<td>PC2</td>
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<td>-</td>
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<td>-</td>
</tr>
<tr>
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<td>2</td>
<td>-</td>
<td>-</td>
<td>( Q_{2}(1 - u_2) )</td>
<td>-</td>
</tr>
<tr>
<td>MB1</td>
<td>none</td>
<td>-</td>
<td>-</td>
<td>( Q_{1} + c_{1} )</td>
<td>-</td>
</tr>
<tr>
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<td>1</td>
<td>( Q_1(1 - u_1) )</td>
<td>-</td>
<td>( r_1 + (Q_{12} - Q_1) + c_2 )</td>
<td>( \alpha \leq 1 )</td>
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<tr>
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<td>( Q_1 + c_{1} )</td>
<td>( Q_1(1 - u_1) )</td>
<td>-</td>
<td>( \alpha \leq 1 )</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>( Q_1(1 - u_1) )</td>
<td>-</td>
<td>( r_1 + (Q_{12} - Q_1)(1 - u_2) )</td>
<td>( \alpha \leq 1 )</td>
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<tr>
<td>MB2</td>
<td>none</td>
<td>-</td>
<td>-</td>
<td>( Q_{2} + c_{2} )</td>
<td>-</td>
</tr>
<tr>
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<td>1</td>
<td>( Q_2(1 - u_1) )</td>
<td>-</td>
<td>( r_2 + (Q_{12} - Q_2)(1 - u_1) )</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( Q_2(1 - u_2) )</td>
<td>( r_2 + (Q_{12} - Q_2)(1 - u_1) )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>( Q_2(1 - u_2) )</td>
<td>( r_2 + (Q_{12} - Q_2)(1 - u_1) )</td>
<td>-</td>
<td>-</td>
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</table>

Table 12: Optimal Selling Prices for a two-component problem with limited capacity
<table>
<thead>
<tr>
<th>Bundling strategy</th>
<th>Component, const. binding</th>
<th>$\lambda_1$</th>
<th>Shadow prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>PB</td>
<td>none</td>
<td>0</td>
<td>$Q_1(\gamma_B - \gamma_1) + Q_2(\gamma_1 - \gamma_2) + Q_B(1 - \gamma_B - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>$Q_1(\gamma_B - \gamma_1) + Q_2(\gamma_1 - \gamma_2) + Q_B(1 - \gamma_B - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>$Q_1(\gamma_B - \gamma_1) + Q_2(\gamma_1 - \gamma_2) + Q_B(1 - \gamma_B - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>$\lambda_1$</td>
<td>$-\lambda_1 + Q_B - c_B - 2Q_Bu_2$</td>
</tr>
<tr>
<td>PC12</td>
<td>none</td>
<td>0</td>
<td>$(Q_1 - Q_2)(1 - \gamma_B - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>$Q_2(1 - (u_1 + u_2))$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>$Q_2(1 - (u_1 + u_2))$</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>$r_2 + (Q_1 - Q_2)(1 - u_1)$</td>
<td>$Q_2(1 - (u_1 + u_2))$</td>
</tr>
<tr>
<td>PC1</td>
<td>none</td>
<td>0</td>
<td>$Q_1(1 - 2u_1) - c_1$</td>
</tr>
<tr>
<td>PC2</td>
<td>none</td>
<td>0</td>
<td>$Q_1(1 - 2u_1) - c_1$</td>
</tr>
<tr>
<td>MB12</td>
<td>none</td>
<td>0</td>
<td>$(Q_1 - Q_2)(1 - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>$(Q_1 - Q_2)(Q_1 - Q_2)(2u_1 - 2u_2 - \gamma_1 + 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>$(Q_1 - Q_2)(Q_1 - Q_2)(2u_1 - 2u_2 - \gamma_1 + 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>$(Q_1 - Q_2)(2u_1 - 2u_2 - 1 - \gamma_1 + \gamma_2 + 2\gamma_B)$</td>
<td>$Q_2(1 - \gamma_B - 2u_2)$</td>
</tr>
<tr>
<td>MB1</td>
<td>none</td>
<td>$2u_2 - 1$</td>
<td>$Q_1(1 - 2u_1) - c_1$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$2u_2 - 1$</td>
<td>$(Q_1 - Q_B)(\gamma_B - 1 + 2u_2)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>$(Q_B - Q_1)(\gamma_B - 2u_2)$</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>$Q_1(1 - \gamma_1 - 2u_1) + Q_2(\gamma_1 - \gamma_2)$</td>
<td>$Q_2(1 - \gamma_B - 2u_2)$</td>
</tr>
<tr>
<td>MB2</td>
<td>none</td>
<td>0</td>
<td>$-(c_B - c_2) + (Q_B - Q_2)(1 - 2u_1)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>$Q_2(1 - \gamma_2 - 2u_2)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>$Q_2(1 - \gamma_2 - 2u_2)$</td>
</tr>
<tr>
<td></td>
<td>1 &amp; 2</td>
<td>0</td>
<td>$Q_2(1 - \gamma_2 - 2u_2)$</td>
</tr>
</tbody>
</table>

where $\gamma_2 = \frac{\gamma_2 B}{Q_2}, \gamma_1 = \frac{\gamma_1 - \gamma_2}{Q_1 - Q_2}$ and $\gamma_B = \frac{\gamma_B - \gamma_1}{Q_B - Q_1}$.

**Table 13:** Shadow prices for a two-component problem with limited capacity
<table>
<thead>
<tr>
<th>strategy</th>
<th>const. binding</th>
<th>Necessary and sufficient conditions</th>
<th>$r_1$</th>
<th>$r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC12</td>
<td>none</td>
<td>$1 - 2u_1 &lt; \frac{c_1}{Q_1 - Q_2} &lt; 1$ and $0 &lt; \frac{c_1}{Q_1 - Q_2} - \frac{c_2}{Q_2} &lt; 2u_2$</td>
<td>$\frac{c_1 + Q_1}{Q_1}$</td>
<td>$\frac{c_2 + Q_2}{Q_2}$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$1 - 2u_2 - 2u_1 &lt; \frac{c_1}{Q_1 - Q_2} &lt; 1 - 2u_1$ and $\frac{c_1}{Q_1 - Q_2} - \frac{c_2}{Q_2} \leq 1 - 2u_1$</td>
<td>$r_2 + (1 - u_1)(Q_1 - Q_2)$</td>
<td>$\frac{c_2 + Q_2}{Q_2}$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$1 - 2Q_1 u_2 - 2u_1 &lt; \frac{c_1}{Q_1} &lt; 1 - 2Q_1 u_2$ and $\frac{c_1}{Q_1} - \frac{c_2}{Q_2} \geq 2u_2$</td>
<td>$r_2 + (Q_1 - Q_2) Q_1 + \frac{c_1 + 2Q_1 u_2}{Q_2}$</td>
<td>$Q_1 - 2Q_1 Q_2 (1 - u_1)$</td>
</tr>
<tr>
<td></td>
<td>1&amp;2</td>
<td>$1 - 2Q_1 u_2 - 2u_1 \geq \frac{c_1}{Q_1}$ and $1 - 2u_2 - 2u_1 \geq \frac{c_2}{Q_2}$</td>
<td>$r_2 + (Q_1 - Q_2) (1 - u_1)$</td>
<td>$(1 - u_1 - u_2)Q_2$</td>
</tr>
<tr>
<td>PC1</td>
<td>none</td>
<td>$1 - 2u_1 &lt; \frac{c_1}{Q_1} &lt; 1$ and $\frac{c_1}{Q_1} \leq \frac{c_2}{Q_2}$</td>
<td>$\frac{c_1 + Q_1}{2}$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\frac{c_1}{Q_1} \leq 1 - 2u_1$ and $\frac{c_2}{Q_2} \geq 1 - 2u_1$</td>
<td>$(1 - u_1)Q_1$</td>
<td>$+\infty$</td>
</tr>
<tr>
<td>PC2</td>
<td>none</td>
<td>$\frac{Q_1 - Q_2}{Q_1} \geq 1$ and $1 - 2u_2 &lt; \frac{c_2}{Q_2} &lt; 1$</td>
<td>$+\infty$</td>
<td>$\frac{c_2 + Q_2}{2}$</td>
</tr>
<tr>
<td>PC2</td>
<td>2</td>
<td>$\frac{Q_1 - Q_2}{Q_1} \leq 1 - 2u_2$ and $\frac{c_1}{Q_1} \geq 1 - 2Q_1 u_2$</td>
<td>$+\infty$</td>
<td>$(1 - u_2)Q_2$</td>
</tr>
</tbody>
</table>

**Table 14:** Selling prices obtained from the IB heuristic for a two-component problem with limited capacity.
<table>
<thead>
<tr>
<th>Bundling strategy</th>
<th>Component const. binding</th>
<th>$r_1$</th>
<th>$r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
</tr>
<tr>
<td>none</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
</tr>
<tr>
<td>none</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12} + (c_1 + c_2)$</td>
</tr>
<tr>
<td>none</td>
<td>$(Q_{12} - Q_{12})(1 - u_1)$</td>
<td>$Q_{1}$</td>
<td>$(Q_{12} - Q_{12})(1 - u_1)$</td>
</tr>
<tr>
<td>PB</td>
<td>$Q_{12}(1 - u_1)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12}(1 - u_1)$</td>
</tr>
<tr>
<td>PC1</td>
<td>$Q_{12}(1 - u_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12}(1 - u_2)$</td>
</tr>
<tr>
<td>PC2</td>
<td>$Q_{12}(1 - u_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12}(1 - u_2)$</td>
</tr>
<tr>
<td>MB1</td>
<td>$Q_{12}(1 - u_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12}(1 - u_2)$</td>
</tr>
<tr>
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<td>$Q_{12}(1 - u_2)$</td>
<td>$Q_{1}$</td>
<td>$Q_{12}(1 - u_2)$</td>
</tr>
</tbody>
</table>

For cases with multiple solutions, we pick the one with the highest expected profit.

$\lambda_1 = \frac{Q_2(Q_1 - Q_2)(c_1 + c_2 + Q_1)}{Q_2(Q_1 - Q_2)(c_1 + c_2 + Q_1)}$

$\lambda_2 = \frac{Q_2(Q_1 - Q_2)(c_1 + c_2 + Q_1)}{Q_2(Q_1 - Q_2)(c_1 + c_2 + Q_1)}$

Table 15: Price for CO heuristic for a two-component problem with limited capacity