Fixed versus Random Proportions Demand Models for the Assortment Planning Problem under Stock-out based Substitution

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We consider the problem of determining the optimal assortment of products to offer in a given product category when each customer is characterized by a type, which is a list of products he is willing to buy in decreasing order of preference. We assume consumer-driven, dynamic, stock-out based substitution and random proportions of each type. No efficient method to obtain the optimal solution for this problem is known to our knowledge. However, if the number of customers of each type is a fixed proportion of demand there exists an efficient algorithm for solving for the optimal assortment. We show that the fixed proportions model gives an upper bound to the optimal expected profit for the random proportions model. This bound allows us to obtain a measure of the absolute performance of heuristic solutions. We also provide a bound for the component-wise absolute difference in expected sales between the two models, which is asymptotically tight as the inventory vector is made large, while keeping the number of products fixed. This result provides us with a lower bound to the optimal expected profit and a performance guarantee for the fixed proportion solution in the random proportion model.

Key words: Assortment planning, Inventory management, Bounds, Stock-out based Substitution  

History:

1. Introduction

An important problem in retailing is that of a firm choosing the assortment of differentiated products to offer when facing customers with heterogenous tastes. Customers’ tastes are characterized by their type, which is a list of products they are willing to buy in decreasing order of preference. The retailer has to decide which products to stock and how much inventory to carry of each of them. The products are offered at different selling prices and have different cost parameters for the retailer. Customers come to the store sequentially, observe the available inventory, and decide which product (if any) to purchase, based on their individual preferences. The customers who come to the store constitute a sample from the entire customer population, so that each customer has
a certain probability of being of each type, independently of the other customers. The retailer’s objective is to maximize expected profit knowing the distribution of the number of customers who visit the store in one period and the distribution of customer preferences in the population, but not the preferences of the specific customers who come to the store. This problem is known as the one-period assortment planning problem with consumer-driven, dynamic, stock-out based substitution with random proportions of customers of each type.

This is a complex and hard problem, see Goyal et al. (2009) who show that it is NP-hard. To our knowledge, no efficient method exists for solving this problem to optimality. Previous work on this problem has focused on providing heuristic methods (e.g., Pentico (1974), van Ryzin and Mahajan (1999), Smith and Agrawal (2000), Mahajan and van Ryzin (2001) and Netessine and Rudi (2003)). Our paper is most closely related to Honhon, Gaur & Seshadri (2010) who assume fixed proportions instead of random proportions, that is, they assume that the proportion of customers of each type who visit the store is fixed and matches the proportion in the entire customer population. They solve the hence modified problem to optimality using a dynamic program approach. For a broader review of the literature, including papers on estimation of demand with dynamic, stock-out based substitution and papers which assume static, assortment-based substitution, we refer to Kok et al. (2009).

Most of the papers do not provide performance bounds for the heuristics they propose (one exception is the work of Gaur and Honhon (2006) who obtain an upper bound based on retailer-controlled substitution for the locational choice model). They compare the relative performance of their heuristic to existing ones but do not provide guidance to distance from optimality. The goal of this paper is to provide an upper bound on the optimal expected profit (Proposition 1), which can be computed quickly (at least for up to $n = 16$ products) and used to calculate the distance from optimality of different heuristics, i.e., to obtain a measure of the absolute performance of the heuristics. Our upper bound is based on the assumption of fixed proportions introduced by Honhon, Gaur & Seshadri (2010). We show that the optimal expected profit under fixed proportions is always greater or equal to the optimal expected profit under random proportions. This upper bound can be easily computed using the dynamic program approach of Honhon, Gaur & Seshadri (2010). We also provide a theoretical bound on the componentwise absolute difference in expected sales between the fixed proportions and random proportions models (Lemma 3), which does not depend on the demand distribution or the parameters of the consumer choice model, is asymptotically tight as demand and inventory become larger, keeping the number of products fixed. We use this result to obtain a lower bound on the optimal expected profit in the random proportions model.
(Proposition 3). The percentage gap between the upper and lower bounds on the optimal expected profit is small when the optimal inventory vector in the fixed proportions model is large and decreases with mean demand and overage costs. Finally we also obtain a performance guarantee for the fixed proportion heuristic (Proposition 2). In our numerical study we find that the average gap between the upper bound and the profit generated by the fixed proportions heuristic is only 0.72%. This gap is particularly small when the mean and variance of demand are large.

The rest of our paper is organized as follows. In Section 2 we briefly describe the one-period assortment planning problem with consumer-driven, dynamic, stock-out based substitution under both random proportions and fixed proportions. Our results are presented in Section 3. The proofs of these results, which are quite involved, are deferred to the online supplement. In Section 4, we present our numerical study. Finally we conclude in Section 5.

2. Model
In this section we briefly describe the random proportions (RP) and fixed proportion (FP) models and introduce the necessary notation. For a more detailed presentation of these models we refer to Mahajan and van Ryzin (2001) and Honhon, Gaur & Seshadri (2010) respectively for the RP and FP model.

We consider a product category consisting of \(n\) potential products, indexed 1 to \(n\). Let \(\mathcal{N} = \{1, \ldots, n\}\). The retailer determines the inventory of each of the \(n\) products at the beginning of the selling season or period. Let \(q_j\) be the inventory of product \(j\) at the start of the period. Let \(\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{N}^n\) be the starting inventory vector. During the period, customers come to the store one after the other and progressively deplete the inventory (dynamic substitution). We assume that each customer buys at most one unit of the product. The retailer cannot replenish or modify the inventory during the period nor can she decide which product to allocate to each customer (consumer-driven substitution). Rather, an arriving customer selects a product from the set of products with positive inventory at the time of their visit to the store or leaves the store without purchasing (stock-out based substitution). Any inventory leftover at the end of the period is salvaged.

Customers are heterogeneous in their choice behavior: each customer belongs to a consumer type, independently of the other customers. A consumer type \(\tau\) is a vector of products that a customer of that type is willing to purchase, arranged in decreasing order of preference. For example, a customer of type \((1, 2, 4)\) has product 1 as his first choice, product 2 as his second choice, product 4 as his third choice, and he never buys products 3 and 5 to \(n\). Let \(\mathcal{T}\) be the set of all possible types.
The retailer does not know the types of the incoming customers. However, she knows how customers make choices and knows the distribution of customer preferences. We assume that the type of a customer is independent of the types of other customers and independent of the total number of customers. Let $\alpha(\tau)$ denote the probability that a customer is of type $\tau$, where $\sum_{\tau \in T} \alpha(\tau) = 1$.

Let $D$ denote the number of customer arrivals during the selling season. $D$ is a non-negative integer-valued finite random variable bounded above by $\overline{D}$.

In the RP model, the number of customers who are of type $\tau$ in a selling season is a random variable with a binomial distribution with $D$ trials and a probability of success of $\alpha(\tau)$. In the FP model, the number of customers of type $\tau$ who come to the store is exactly $D \alpha(\tau)$. Moreover, out of the first $k$ customers who come to the store, there are exactly $k \alpha(\tau)$ customers of type $\tau$ for all $0 \leq k \leq D$. In essence, demand is continuous and follows a fluid model: every unit of demand divides exactly into $\alpha(\tau)$ customers of type $\tau$ for all possible types.

Let $\omega$ denote a generic sample path. In the RP model, a sample path is a sequence of customer arrivals along with their types. In the FP model, there is only one source of uncertainty, which is the demand $D$, therefore a sample path is simply a value for $D$. The expectations are taken over all sample paths in both models accordingly.

The superscripts $R$ and $F$ are appended to the following set of variables in order to denote random and fixed proportions respectively. Let $x(k, q; \omega)$ denote the leftover inventory vector on sample path $\omega$ after $k$ customers have depleted the inventory when the starting inventory is $q$. Let $y(q; \omega)$ denote the sales inventory vector on sample path $\omega$ when the starting inventory is $q$ after all customers on sample path $\omega$ have arrived. The one-period profit for starting vector $q$ on sample path $\omega$ is $\Pi(q; \omega) = \sum_{j=1}^{n} (u_j + o_j) y_j(q; \omega) - \sum_{j=1}^{n} o_j q_j$ where $u_j$ and $o_j$ are respectively the underage cost and overage cost of product $j$. The retailer’s objective is solve $\max_{q \geq 0} E[\Pi(q)]$. Let $q^{*R}$ and $q^{*F}$ denote the optimal inventory vector in the RP and FP model respectively.

Mahajan and van Ryzin (2001) show that solving for $q^{*R}$ is very difficult because the objective function is not quasi-concave in inventory levels. To the best of our knowledge, finding the optimal solution $q^{*R}$ in an efficient way is still an open question. For small values of $n$ and $\overline{D}$, one could resort to an exhaustive search over all integer-value inventory vectors such that $0 \leq q_j \leq \overline{D}$ for $j = 1, \ldots, n$, however this is too time-consuming for most problems. Therefore, $q^{*R}$ is generally unknown.

Honhon, Gaur & Seshadri (2010) provide an efficient dynamic programming-based algorithm to obtain $q^{*F}$. This algorithm has a theoretical complexity of $O(8^n)$ but this number is based on a worst case analysis of the number of breakpoints in the value functions (which is equal to $2^{3n-1} + 2^{2n-1}$). In practice, the number of breakpoints encountered is usually much lower so that
the algorithm is very fast: we were able to obtain \( q^*F \) in less than 12 minutes for up to 16 products and with an average of approximately 65,000 breakpoints (for \( n = 16 \) the upper bound on the number of breakpoints is greater than \( 10^{13} \)).

3. Results

In this section we show that the FP model provides an upper bound on the optimal expected profit \( \mathbb{E}[\Pi_R(q^*R)] \) in the RP model. This upper bound can be used to estimate optimality gaps when testing the absolute performance of heuristics.

First note that, for a given vector \( q \) it is possible that the expected profit under random proportions is greater than the expected profits under fixed proportions, i.e., \( \mathbb{E}[\Pi_F(q)] < \mathbb{E}[\Pi_R(q)] \), as the next example illustrates.

**Example 1.** Let \( n = 2 \). There are 4 possible types: \( T = \{(1), (2), (1, 2), (2, 1)\} \). All types are equally likely, i.e., \( \alpha(\tau) = 1/4 \) for all \( \tau \in T \). Let \( q = (2, 1) \). Demand is deterministic and \( D = 2 \). Let \( u_1 = 10, u_2 = 1, o_1 = 1, o_2 = 3 \). We have \( \mathbb{E}[\Pi_F(q)] = 10 < \mathbb{E}[\Pi_R(q)] = 10.375 \).

However, we have the following result:

**Lemma 1.** For every \( q \geq 0 \) there exists \( \tilde{q} \geq 0 \) such that \( \mathbb{E}[\Pi_F(\tilde{q})] \geq \mathbb{E}[\Pi_R(q)] \).

We provide here a sketch of the proof, which is fairly involved. The general idea of the proof is that we are able to match the expected profit from a fixed quantity under random proportions with the expected profit from a random quantity under fixed proportions. Because there is no randomness with regard to the relative number of customers of each type in the fixed proportion model, we introduce randomness in the starting inventory vector in order to match the randomness in the random proportion model. The details are as follows. Given a vector \( q \) for the RP system, for each sample path \( \bar{\omega} \) such as \( D(\bar{\omega}) = \mathcal{D} \), we select a starting inventory vector to be used in the FP model in such a way that the two models have the same set of products with positive inventory for all values of demand and the same leftover inventory vector after \( D \) customers have come. The FP system is started randomly with these vectors with probability \( P(\bar{\omega}|D(\bar{\omega}) = \mathcal{D}) \), which add up to one. We show that the weighted average of the expected sales in the FP model with these starting inventory vectors matches the expected sales in the RP model and that the same holds for expected profit. As a result there must exist an inventory vector such that expected profit in the FP model is higher than \( \mathbb{E}[\Pi_R(q)] \). In the Appendix we illustrate the proof technique with an numerical example.

Lemma 1 reveals a startling duality between the random proportions and the fixed proportions model, namely, that the fixed proportions model is equivalent to the random proportions model.
under a properly chosen random starting vector. We conjecture that such an equivalence can be established even when inventory is replenished during a sales campaign.

This leads to our main proposition:

**Proposition 1.** $E[\Pi^F(q^F)] \geq E[\Pi^R(q^R)]$.

Using Lemma 1 we can find $\tilde{q}$ such that $E[\Pi^F(\tilde{q})] \geq E[\Pi^R(q^*)]$. The result follows by the optimality of $q^*$. Q.E.D.

Remember that, while $E[\Pi^R(q^*)]$ is generally unknown, $E[\Pi^F(q^*)]$ can be computed as shown in Honhon, Gaur & Seshadri (2010). It follows that the performance of a heuristic solution $q$ for the RP model can be evaluated with respect to $E[\Pi^F(q^*)]$, i.e., we can compute $E[\Pi^F(q^*)] - E[\Pi^R(q^*)]$, which is an upper bound on the optimality gap $E[\Pi^R(q^*)] - E[\Pi^R(q)]$.

The next two results show that the FP model does not over-estimate component-wise expected sales by too much, proving that the difference between $E[\Pi^F(q^*)]$ and $E[\Pi^R(q^*)]$ is generally small. First Lemma 2 provides a bound on the difference in sales of one product up to the first stock-out epoch between the FP and RP models.

**Lemma 2.** Given $q \geq 0$, suppose that $D = k \leq \min_{j=1,...,n} \frac{q_j}{\rho_j(q)}$ where $\rho_j(q)$ is the probability that a customer chooses product $j$ given inventory vector $q$. We have $(E[y^F_j(q)|D = k] - E[y^R_j(q)|D = k])^+ \leq \frac{\sqrt{2\pi}}{2\sqrt{k}}$ for $j = 1, ..., n$.

To prove Lemma 2 we consider the extreme case in which customers do not substitute in the event of a stock-out in the RP model and maximize the component-wise difference in the expected sales between this case and the FP model. We use Lemma 2 to obtain a bound on the sum of component-wise difference in expected sales between the two models:

**Lemma 3.** Given $q \geq 0$, $\sum_{j=1}^n (E[y^F_j(q)] - E[y^R_j(q)])^+ \leq \frac{\sqrt{2\pi}}{\sqrt{Q}} \sum_{j=1}^n \sqrt{q_{[j]}}$ where $q_{[j]}$ is the $j$-th greatest value in $q_1, ..., q_n$. Also if $Q = \sum_{j=1}^n q_j$, we have:

$$\sum_{j=1}^n (E[y^F_j(q)] - E[y^R_j(q)])^+ \leq \frac{\sqrt{n(n+1)}}{\sqrt{\pi} \sqrt{Q}}.$$  

In the proof of Lemma 3, we compare the FP and RP models for the same starting inventory $q$. We use Lemma 2 to bound the difference in expected sales up to the first stock-out epoch in the FP model. Then, we carefully adjust the leftover inventory vectors at that stock-out epoch in the two models so that they match. When adjusting the leftover inventory in the RP model down, we take into account the fact that this decrease may increase the future sales of other products. Then, we use Lemma 2 to bound the difference in expected sales between the first and second stock-out
epochs in the FP model and repeat the adjustment process. We do so until the very last product runs out in the FP model.

From (1), we see that the bound is asymptotically tight as the inventory vector grows larger and the number of items in the assortment is fixed.

**Example 2.** For \( n = 3 \) and \( Q = 300 \) the right-hand side of (1) is 11.28%. If \( Q = 3000 \), it is 3.57%.

We use Lemma 3 to obtain a theoretical bound on the performance of the \( q^*F \) heuristic in the RP model:

**Proposition 2.** \( \mathbb{E}[\Pi^R(q^*F)] \geq \mathbb{E}[\Pi^F(q^*F)] - (\max_{j=1,\ldots,n}(u_j + o_j)) \left( \frac{\sqrt{\pi}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{jq^*F[j]} \right) \), where \( q^*F[j] \) is the \( j \)-th greatest value in \( q^*F_1,\ldots,q^*F_n \).

Also we directly obtain a lower bound on the expected profit in the RP model:

**Proposition 3.** \( \mathbb{E}[\Pi^R(q^*R)] \geq \mathbb{E}[\Pi^F(q^*F)] - (\max_{j=1,\ldots,n}(u_j + o_j)) \left( \frac{\sqrt{\pi}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{jq^*F[j]} \right) \), where \( q^*F[j] \) is the \( j \)-th greatest value in \( q^*F_1,\ldots,q^*F_n \).

The gap between this lower bound and the upper bound of Proposition 1, i.e., \( (\max_{j=1,\ldots,n}(u_j + o_j)) \left( \frac{\sqrt{\pi}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{jq^*F[j]} \right) \), increases with \( q^*F \) in absolute terms however it decreases as a proportion of the upper bound because of the square root term. An increase in the optimal inventory vector can be caused either by an increase in mean demand or by an increase in the overage cost. In both cases the gap becomes smaller in relative terms as illustrated in the next section. In the case of an increase in mean demand, the explanation is that the coefficient of variation of the proportion of customers from each type, which is equal to \( \sqrt{\mathbb{E}[\frac{1}{D}] - \alpha(\tau)} \), decreases with the mean \( \mu \), which leads to the RP model being closer to the FP model (because \( 1/D \) is a decreasing convex function, \( \mathbb{E}[1/D] \) gets smaller as \( D \) becomes stochastically larger (or more generally smaller in convex order). In the case of an increase in the overage cost, the higher inventory levels lead to fewer stock-outs, so that the FP model becomes a better approximation of the RP model.

Note that the bounds we obtain are free of any assumption on the distribution of demand and the preference structure (i.e., on the \( \alpha_\tau \) for \( \tau \in \mathcal{T} \)). In such they apply to a very wide range of problems and settings since most consumer choice models used in the operations literature are special cases of ours. It may be possible to improve the quality of the bounds by making them model-specific and dependent on the demand distribution (e.g., in Lemma 3 one could multiply the upper bound on the expected difference in sales by the probability of the first run out, the second run out, etc.). Our current focus was to demonstrate the use of the fixed proportion model in all generality so we decided not to do so and leave this extension for our future research plans.
4. Numerical study

The objective of this section is threefold. First we study how well the FP model approximates the expected sales obtained with the RP model and compare the FP model with another fluid-model type approximation, namely the one suggested by Hopp and Xu (2008). Second, we study how the percentage gap between the upper bound from Proposition 1 and the lower bound from Proposition 3 varies with some key parameters of the model. Finally, we compare three heuristics in terms of speed and absolute performance using the upper bound from Proposition 1.

Study of the quality of the FP approximation

Hopp and Xu (2008) propose a static approximation of the assortment planning problem with consumer-driven, dynamic, stock-out based substitution which is based on a fluid network model and a one-to-one mapping of service levels and inventory quantities. In what follows, we refer to their model as the HX model. Like in the FP model, the HX model ignores the randomness in the number of customers from each type and assumes that the demand is continuous. But the main differences with the FP model are that (a) the HX model only applies to a consumer choice model which is an attraction-type model, (b) in calculating the service level values corresponding to the inventory levels, the HX model assumes that the probability of switching from product $i$ to product $j$ following a stock-out of product $i$ is equal to the probability of choosing product $j$ as a first choice. This simplification, which the authors refer to as the memoryless flow assumption, leads to an overestimation of demand because a proportion of unmet demand gets sent back to the original product requested.

Given that an attraction-based model like the Multinomial Logit (MNL) model is a special case of the ranking-based model we consider in this paper (in the sense that is possible to define $\alpha(\tau)$ for $\tau \in \mathcal{T}$ such that the choice probabilities of any assortment exactly match those obtained with the MNL model), we are able to directly compare the HX, FP and RP model in terms of expected sales for a number of examples. We use the same parameters as in Example 2 from Hopp and Xu (2008): we assume that the number of customers visiting the store is normally distributed with mean $\mu = 1000$ and standard deviation $\kappa \sqrt{\mu}$ and vary $\kappa \in \{0.25, 0.5, 0.75, 1, 2, 4, 6, 8\}$. We assume $n = 3$ and let $\mathbf{v} = (v_0, v_1, v_2, v_3)$ be the vector of nominal utilities from the MNL model, i.e., such that the probability of picking product $j$ from set $S$ is given by $\frac{v_j}{v_0 + \sum_{i \in S} v_i}$. We fix the inventory level $\mathbf{q}$ and calculate the expected sales with the HX and FP models exactly. Let $y_{HX}^{j}$ denote sales in the HX model. Similarly to Hopp and Xu (2008), we use simulation to obtain expected sales under the RP model. For each problem instance we report the percentage error of the two approximations, calculated as $\frac{\mathbb{E}[y_{RP}^j(\mathbf{q})] - \mathbb{E}[y_{HX}^j(\mathbf{q})]}{\mathbb{E}[y_{RP}^j(\mathbf{q})]}$ and $\frac{\mathbb{E}[y_{RP}^j(\mathbf{q})] - \mathbb{E}[y_{RP}^j(\mathbf{q})]}{\mathbb{E}[y_{RP}^j(\mathbf{q})]}$ for $j = 1, 2, 3$, respectively for the FP and HX model.
models. Like Hopp and Xu (2008), we divide the problem instances into four scenarios; Table 1 is constructed after their Table 1 on page 636. The percentage errors we obtain for the HX model do not exactly match those from Table 1 in Hopp and Xu (2008) because they are based on simulated data; however they are always very close.

Table 1 Percentage errors in expected sales of the three products for the HX and FP models

<table>
<thead>
<tr>
<th>Scenario 1 (%)</th>
<th>Scenario 2 (%)</th>
<th>Scenario 3 (%)</th>
<th>Scenario 4 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v = (1,1,1,1)$</td>
<td>$v = (1,1,1,1)$</td>
<td>$v = (1,1,2,3)$</td>
<td>$v = (1,1,2,3)$</td>
</tr>
<tr>
<td>$q = (250,250,250)$</td>
<td>$q = (150,300,450)$</td>
<td>$q = (150,300,450)$</td>
<td>$q = (250,250,250)$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$HX$</td>
<td>$FP$</td>
<td>$HX$</td>
</tr>
<tr>
<td>0.25</td>
<td>0.9090909</td>
<td>0.030-0.1</td>
<td>0.030-0.1</td>
</tr>
<tr>
<td>0.50</td>
<td>0.7070707</td>
<td>0.030-0.2</td>
<td>0.030-0.2</td>
</tr>
<tr>
<td>0.75</td>
<td>0.6060606</td>
<td>0.010-0.2</td>
<td>0.040-0.2</td>
</tr>
<tr>
<td>1.00</td>
<td>0.5050505</td>
<td>0.010-0.3</td>
<td>0.050-0.3</td>
</tr>
<tr>
<td>2.00</td>
<td>0.3030304</td>
<td>0.041-0.5</td>
<td>0.050-0.2</td>
</tr>
<tr>
<td>4.00</td>
<td>0.2030302</td>
<td>0.1721-1.2</td>
<td>0.040</td>
</tr>
<tr>
<td>6.00</td>
<td>0.2030303</td>
<td>4.233-1.8</td>
<td>0.1040.2</td>
</tr>
<tr>
<td>8.00</td>
<td>0.2030303</td>
<td>7.842-2.7</td>
<td>0.101-0.4</td>
</tr>
</tbody>
</table>

The main observation is that, while both approximations have a tendency to overestimate expected sales, the FP model is always more accurate. Further, the performance of the FP model improves as the variance of demand increases while that of the HX model deteriorates. Our interpretation of this seemingly counter-intuitive result is as follows: when the number of customers coming to the store is more variable, situations with no product stocking out and situations with every product selling out become more likely. In these two extreme cases, sales under the FP model exactly match sales under the RP model. Differences between the two models occur when only a few products stock-out and this is more likely to happen when the variance of demand is low. The results from Table 1 suggests that the FP model is a more accurate approximation of the RP model than the HX model, especially when the variance of demand is large. Our conclusion is that the extra simplification made by the HX model (i.e., the memoryless flow assumption) results in a more tractable formulation for expected sales but also in a deterioration of the quality of the approximation.

Study of the gap between lower and upper bounds

Next we study the percentage gap between the lower bound and the upper bound, measured by $\left(\max_{j=1,...,n}(u_j+\sigma_j)\sqrt{\sum_{j=1}^{n}\sigma_j}\sqrt{\gamma_{HF}}\right)\times 100$. We assume preferences are modeled by the MNL model with vector of nominal utilities $v = (v_0, v_1, ..., v_n) = (5,1,2,3,...,n)$. We let $u_j = \sigma_j = 5$ for $j = 1,...,n$. We assume that demand has a Normal distribution with mean $\mu$ and standard deviation $\kappa\sqrt{\mu}$. We vary the number of products $n \in \{8,12,16\}$, $\mu \in \{1000,5000,10000,50000\}$ and $\kappa \in \{0.25,0.5,0.75,1,2,4,6,8\}$. 
In Figure 1 we see that the percentage gap between the lower and upper bound decreases with $\mu$ but increases slightly with $\kappa$ and $n$. For $\mu = 50,000$ and $\kappa = 8$ the average percentage gap is equal to 3.29%. Remember that this percentage gap is an upper bound on the distance between the upper bound and the optimal expected profit and is based on a worst case analysis which does not include the demand distribution and is based on a re-balancing of the inventory vector when a run-out happens. Hence these numbers are very encouraging and suggest that the upper bound is good especially for large problems.

In our numerical investigations, we also found that the gap between the lower and upper bound increases almost linearly with the maximum overage and underage cost value, which is as expected since the difference between the lower and upper bound is $(\max_{j=1,\ldots,n}(u_j + o_j)) \left( \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{j q^*_{[j]}} \right)$.

**Study of the performance of heuristics**

Next, we study the performance of $q^*F$ as a heuristic for the random proportions model by comparing it to two previously studied heuristics, namely the assortment-based substitution (ABS) heuristic and the Sample Path Gradient Algorithm (SPGA) from van Ryzin and Mahajan (1999) (the solution method suggested by Hopp and Xu (2008) does not apply here because we assume prices are fixed). Such a comparison was also done by Honhon, Gaur & Seshadri (2010), however, because the authors were unaware of the results from this paper and because the optimal solution, i.e., $q^*R$, was unknown in all the problem instances they considered, all they could do was compare the relative performance of the three heuristics. Thanks to Proposition 1, we are now able to provide an upper bound on the distance from optimality for each heuristic, that is, measure the
absolute performance of the three heuristics.

The methodology used is as follows. We compute $q^F$ using the dynamic programming algorithm of Honhon, Gaur & Seshadri (2010) then we round the inventory values to the nearest integer. We refer to Honhon, Gaur & Seshadri (2010) and van Ryzin and Mahajan (1999) for an explanation of how to compute $q^A$ and $q^S$ which are respectively the solutions of the assortment-based substitution model and Sample Path Gradient Algorithm. For each heuristic solution $q$, we estimate $\Pi^R(q;\omega)$ and $E[\Pi^F(q^F)]$ by simulation using 10,000 sample paths of random customer arrivals.

For each problem instance, we plot the percentage gap relative to the bound, computed as 

$$\frac{E[\Pi^F(q^F)] - E[\Pi^R(q)]}{E[\Pi^F(q^F)]} \times 100$$

where $q$ is equal to $q^F$, $q^A$ and $q^S$. We present our results in four scenarios (which are inspired by those used in Honhon, Gaur & Seshadri (2010)). We compute the standard error of the paired difference in profit (for each pair of heuristics) over the 10,000 replications. Based on this we find that the difference in the average profit over the 10,000 replications is significant at the 1% level in 380 out of 405 cases. Specifically, FP outperforms ABS 89 times and ABS outperforms FP 25 times. In 21 experiments their performance is statistically indistinguishable. FP outperforms SPGA 82 times and is outperformed 50 times by SPGA with 3 outcomes being indistinguishable. ABS outperforms SPGA 78 times and is outperformed 56 times by SPGA with 4 ties. We elaborate upon when each algorithm performs the best below. Because the average difference in profit is significantly different from zero in 94% of the cases, we report just the average profit over the replications and do not plot the confidence intervals.

In Scenario 1, we study the impact of the mean and variance of demand on the performance of the three heuristics. We set $n=5$ and use a MNL model with vector of nominal utilities $v = (v_0, v_1, ..., v_n) = (1,...,1)$. We let $u_j = 4.5$ for $j = 1,...,5$ and let $o = (8.5, 7, 5.5, 4, 2.5)$. We assume that demand has a Normal distribution with mean $\mu$ and standard deviation $\kappa\sqrt{\mu}$ and vary $\mu \in \{1000, 2000, ..., 5000\}$ and $\kappa \in \{0.25, 0.5, 0.75, 1, 2, 4, 6, 8\}$. In Figure 2 we see that overall, the FP heuristic performs the best, beating the SPGA heuristic always, and the ABS heuristic for high values of $\kappa$. We also see that the FP heuristic performs better as $\mu$ and $\kappa$ increase. The finding with regard to $\kappa$ is consistent with our finding that the FP model becomes a more accurate approximation to the RP model when the variance of demand increase. The improvement with $\mu$ can be explained by the fact that the coefficient of variation of the proportion of customers of a given type, which is equal to $\sqrt{\frac{\mu}{o\alpha(\tau)}}$, is decreasing in $\mu$ (given the demand distribution we use), so that the FP model tends to mimic the RP model as mean demand increases. On the other hand, the ABS heuristic performs worse as $\kappa$ increases. In order to save on inventory costs, the ABS heuristic stocks fewer products when $\kappa$ is really high but this results in a fewer opportunities
for stock-out based substitutions, which affects sales and expected profits. The deterioration of the performance of the SPGA as $\mu$ increases can be explained by the fact that the algorithm needs more iterations to converge to a good solution when the problem size is larger.

![Figure 2 Percentage gaps with respect to upper bound as a function of $\mu$ and $\kappa$ (Scenario 1).](image)

In Scenario 2, we study the impact of the number of products on the performance of the three heuristics. We use a MNL model with vector of nominal utilities $\mathbf{v} = (v_0, v_1, ..., v_n) = (1, ..., 1)$. We let $u_j = 2 + 0.5j$ and $o_j = 7 - 0.5j$ for $j = 1, ..., n$. We assume that demand has a Normal distribution with mean and variance 1000. We vary $n \in \{5, 6, ..., 10\}$. In Figure 3, we see that the FP heuristic performs better as $n$ increases. This can again be explained by the fact that the coefficient of variation of the proportion of customers of a given type decreases as $n$ increases since $\alpha(\tau)$ decreases: the average value of $\alpha(\tau)$ goes from 41/10000 when $n = 5$ to 92/100,000,000 when $n = 10$. The SPGA tends to perform worse as the number of products increases because it takes longer for the algorithm to converge to a solution.

In Scenario 3, we study the impact of the overage cost (value and asymmetry) on the performance of the three heuristics. We set $n = 5$ and use a MNL model with vector of nominal utilities $\mathbf{v} = (v_0, v_1, ..., v_n) = (1, ..., 1)$. We assume that demand has a Normal distribution with mean and variance 1000. We let $u_j = 4.5$ for $j = 1, ..., 5$ and vary $o = (o_1, ..., o_n)$ in two ways. First we let $o_j = k$ for $j = 1, ..., n$ and vary $k \in \{0.5, 1.5, ..., 35.5\}$. This has the effect of varying the sum of overage and underage cost from 5 to 40 and the critical fractiles from 0.11 to 0.9, while keeping the product category homogenous in terms of costs and popularity. Figure 4(left) shows the average
percentage gaps of all three heuristics as a function of $u_1 + o_1$. We see that the performance of the ABS heuristic deteriorates as the overage cost increases. A comparison of the inventory vectors reveals that the ABS tends to understock the products because it ignores the benefits of dynamic substitution. The SPGA initially performs badly for very high critical fractiles then it becomes the best of all three heuristics. The gap of the FP heuristic is increasing in the overage cost but always remains low.

Second, we let $o = (0.5 + 4k, 0.5 + 3k, 0.5 + 2k, 0.5 + k, 0.5)$ and vary $k \in \{0, 0.25, \ldots, 8.75\}$. This has the effect of varying the maximum sum of overage plus underage cost varies between 5 and 40 and the critical fractiles from 0.11 to 0.9, while progressively making the product cost parameters more and more asymmetric. Figure 4(right) shows the average percentage gaps of all three heuristics as a function of $u_1 + o_1$. We see that performance of the three heuristics is similar as in the previous case: the SPGA initially performs badly, then it becomes the best heuristic and the ABS and FP heuristics perform worse as the cost parameters become more asymmetric. Note that Mahajan and van Ryzin (2001) also remarked that the ABS heuristic had worse performance when variants have highly different profit margins.

In Scenario 4, we study the impact of the heterogeneity in consumer preferences on the performance of the three heuristics. We set $n = 5$ and assume that demand has a Normal distribution with mean and variance 1000. We let $u_j = 4.5$ for $j = 1, \ldots, 5$ and $o = (8.5, 7, 5.5, 4, 2.5)$. We use a MNL model with vector of nominal utilities $v$ which we vary as follows: we start initially from $(v_1, \ldots, v_n) = (5, \ldots, 5)$ then progressively shift the popularity to the first product, creating heterogeneity in customer preferences: the next vectors are $(6, 5, 5, 5, 4)$, followed by $(7, 5, 5, 4, 4)$ etc. until

Figure 3 Percentage gaps with respect to upper bound as a function of the number of products (Scenario 2).
(21, 1, 1, 1, 1). In all cases we use $v_0 = 5$. Figure 5 shows the percentage gaps of the three heuristics as a function of $v_1$. We see that the ABS and FP heuristics are very close to one another and they both perform much better than the SPGA heuristic, which cannot handle very heterogeneous product categories in terms of popularity. This is because the SPGA starts with an initial solution where all products are stocked in equal quantities, so it takes longer for the algorithm to converge when the optimal inventory vector is highly asymmetric.

We now discuss the computation (CPU) time required to obtain the solution under the three heuristics. We varied $\mu \in \{1000, 2000, ..., 5000\}$ and $n \in \{10, 11, ..., 16\}$ (we used a Lenovo Thinkpad X201T series laptop with Intel Core i7 CPU L 620 @ 2.00 GHz and 4.00GB of RAM). Our results
indicate that the ABS heuristic is the fastest, followed by the FP heuristic, followed by the SPGA. The CPU time of the SPGA increases rapidly with mean demand (from about 6 minutes with $\mu = 1000$ to about 30 minutes with $\mu = 5000$) but this is not the case for the other two heuristics, for which $n$ is the main factor determining the speed. The CPU time of the ABS heuristic was under 1 minute in all the cases we considered. The FP heuristic is quick for up to 16 products (less than 12 minutes).

Across all four scenarios, the average percentage gaps relative to the upper bound of each heuristic were equal to 0.72%, 1.79% and 9.10% respectively for the FP, ABS and SPGA heuristics, which suggests that the FP heuristic performs the best of all three, on average. The ABS heuristic does not perform well when demand has a high variance and overage costs are high or highly asymmetric. The SPGA is not to be used when the product critical fractiles are high and products are heterogeneous in terms of popularity. Further the SPGA becomes impractical with large values of demand as it requires a very large number of iterations in order to converge to a good solution. For lower values of $n$ (up to 16), when the computational time of the FP heuristic is not significantly larger than that of the ABS heuristic, we recommend the use of the FP heuristic in most cases, except if the overage costs are highly asymmetric and mean demand is low (in which case the SPGA performs better) or if demand has low variance and the cost parameters are low and not too asymmetric (in which case the ABS performs better). Overall, the FP heuristic performs extremely well especially when $n$ is small and the mean and variance of demand are large, which is the case for high volume product categories with a relatively small assortment breadth, such as milk or sodas.

5. Conclusion

To our knowledge, our paper is the first paper to provide a good and computable upper bound on the optimal expected profit for the one-period assortment planning problem with consumer-driven, dynamic, stock-out based substitution with random proportions of customers of each type and a very general model of consumer choice. This upper bound can be used to measure the absolute performance of heuristics in the random proportion model. We show analytically and numerically that the gap between this upper bound and the optimal expected profit is generally small.

Also, we provide a performance guarantee for the fixed proportion heuristic in the random proportion model and further numerical evidence of its good performance.

We hope that our work will be of use to researchers who want to propose new heuristic solutions for this problem and want to test the performance of their method. Also, our proof techniques for establishing the bounds might be of independent interest to researchers who use fluid approximations to dynamic stochastic inventory systems.
References


Online Supplement: Proofs

We introduce the following notations which are useful for the proofs below. Let $e^{j-k}$ for $k \geq j$ denote the $n \times 1$ vector with zeros everywhere except 1 in components $j$ to $k$. Let $I^R(q; \omega)$ be the number of stock-outs on sample path $\omega$ if the starting inventory vector is $q$ in the RP model. Let $t^R(i, q; \omega)$ denote the value of demand at which the $i$-th stock-out occurs in the RP model when the starting inventory is $q$ on sample path $\omega$. Let $t^R(0, q; \omega) = 0$ for all $\omega$. Let $t^F(i, q)$ denote the value of
demand at which the $i$-th stock-out would occur in the FP model if demand were infinite. Let $f(\tau, q)$ denote the $(n \times 1)$ vector which represents the choice of a customer of type $\tau = (\tau_1, \ldots, \tau_m)$ when faced with inventory vector $q \in \mathbb{N}^n$.

**Proof of Lemma 1**

We prove the lemma using two claims.

**Claim 1:** For every sample path $\omega$ such that $D(\omega) = \bar{D}$, there exists $q^\bar{x}$ such that the set of products with positive inventory is the same before each customer arrival if we start the RP model with inventory vector $q$ or if we start the FP model with inventory vector $q^\bar{x}$. Moreover, the leftover inventory vector after $\bar{D}$ customers have come is identical, i.e., $x^R(D, q; \omega) = x^F(D, q^\bar{x})$.

Construct $q^\bar{x}$ in the following way:

$$q^\bar{x} = x^R(D, q; \omega) + \sum_{i=1}^{t_{R}(q; \omega)} \rho(x^R(tR(i-1, q; \omega), q; \omega)) (tR(i, q; \omega) - tR(i-1, q; \omega))$$

$$+ \rho(x^R(tR(i, q; \omega), q; \omega)) (\bar{D} - tR(q; \omega), q; \omega)). \quad (2)$$

In other words, $q_j^\bar{x}$ for $j = 1, \ldots, n$ is obtained by adding up the quantities that would be depleted in the FP model between each pair of stock-out epochs on sample path $\omega$ in the RP model.

**Proof of Claim 1:** Let $\hat{k} = I^R(q; \omega)$, i.e., the number of stock-outs on sample path $\omega$ under RP. Without loss of generality let us assume that the products are numbered such that products $1, \ldots, \hat{j}$ have zero inventory in $q$, the $i$-th product to stock out is product $\hat{j} + i$ for $i = 1, \ldots, \hat{k}$ and the products which have positive leftover inventory after $\bar{D}$ customers have come are products $\hat{j} + \hat{k} + 1, \ldots, n$. In this case, we get from (2) that

$$q_l^\bar{x} = \begin{cases} 0 & l = 1, \ldots, \hat{j}, \\ \sum_{i=1}^{l-\hat{j}} \rho_i(e_j+1, n) (tR(i, q; \omega) - tR(i-1, q; \omega)) & l = \hat{j} + 1, \ldots, \hat{j} + \hat{k}, \\ \ldots \\ \sum_{i=1}^{l-\hat{j}} \rho_i(e_{\hat{j}+1}, n) (tR(i, q; \omega) - tR(i-1, q; \omega)) + \rho_{\hat{j}+1}(e_{\hat{j}+1}, n) (\bar{D} - tR(k, q; \omega)) + x_R^F(D, q; \omega) & l = \hat{j} + \hat{k} + 1, \ldots, n. \end{cases}$$

Hence, we have $\{j = 1, \ldots, n : q_j > 0\} = \{\hat{j} + 1, \ldots, n\}$, that is, the set of products with positive inventory is the same in $q$ and $q^\bar{x}$.

We have

$$tF(1, q^\bar{x}) = \min_j \frac{q_j^\bar{x}}{\rho_j(q^\bar{x})} = \min_j \frac{q_j^\bar{x}}{\rho_j(e_j+1, n)} = \frac{q_j^\bar{x}}{\rho_1(e_1+1, n)} = tF(1, q; \omega).$$

and therefore $\{j = 1, \ldots, n : x_j^F(tF(1, q^\bar{x}), q; \omega) > 0\} = \{j = 1, \ldots, n : x_j^F(tF(1, q^\bar{x}), q) > 0\} = \{\hat{j} + 2, \ldots, n\}$. In other words the first stock-out happens at the same time in the FP and RP models and it is product $\hat{j} + 1$ that runs out first in both models, so that the set of products with positive
 which proves Claim 1. In words, the expected leftover inventory vector starting with \( q \) and in the FP model starting with \( q^D \). After the first stock-out, the leftover inventory in the FP model is:

\[
x^F_i(t^F(1, q^D), q^D) = \begin{cases} 
0 & l = 1, \ldots, j + 1, \\
\sum_{i=2}^{l_1} \rho_i(e^{j+i,n}) (t^R(i, q; \omega) - t^R(i-1, q; \omega)) & l = j + 2, \ldots, j + k, \\
\sum_{i=2}^{l_1} \rho_i(e^{j+i,n}) (t^R(i, q; \omega) - t^R(i-1, q; \omega)) + \rho_i(e^{j+k+1,n}) (\bar{D} - t^R(k, q; \omega)) + x^R(D, q; \omega) & l = j + k + 1, \ldots, n.
\end{cases}
\]

By repeating a similar argument, we can prove that \( t^F(i, q^D) = t^R(i, q; \omega) \) and \( \{j = 1, \ldots, n : x^F_j(t^R(i, q, \omega), q; \omega) > 0\} = \{j = 1, \ldots, n : x^F_j(t^F(i, q^D), q) > 0\} = \{j + i + 1, \ldots, n\} \), for \( i = 1, \ldots, \hat{k} \). Then, after the last stock-out has occurred, the leftover inventory in the FP model is:

\[
x^F_j(t^F(\hat{k}, q^D), q^D) = \begin{cases} 
0 & l = 1, \ldots, \hat{j} + \hat{k}, \\
\rho_i(e^{\hat{j}+k+1,n}) (\bar{D} - t^R(\hat{k}, q; \omega)) + x^R(D, q; \omega) & l = \hat{j} + \hat{k} + 1, \ldots, n.
\end{cases}
\]

Finally we find the inventory vector after the last customer has come using (??):

\[
x^F(\bar{D}, q^D) = x^F(\bar{D}, q^D) - \rho\left( x^F(t^F(\hat{k}, q^D), q^D) \right) \left[ \bar{D} - t^F(\hat{k}, q^D) \right], \\
x^F = \begin{cases} 
0 & l = 1, \ldots, \hat{j} + \hat{k}, \\
x^R(\bar{D}, q; \omega) & l = \hat{j} + \hat{k} + 1, \ldots, n,
\end{cases}
\]

(3) (4)

which proves Claim 1.

**Claim 2:** For \( k \in \{0, \ldots, \bar{D}\} \),

\[
E[x^R(k, q)|\mathcal{D} = k] = \sum_{\omega: D(\omega) = \bar{D}} P(\omega|D = \bar{D}) x^F(k, q^D).
\]

In words, the expected leftover inventory vector starting with \( q \) in the RP model after \( k \) customers have come is equal to the weighted average of the leftover inventory vectors starting with \( q^D \) in the FP model after \( k \) customers have come, where the weights are equal to the probability of occurrence of each sample path.

**Proof of Claim 2:** We do so by induction on \( k \). First, for \( k = \bar{D} \), we have

\[
E[x^R(\bar{D}, q)|\mathcal{D} = \bar{D}] = \sum_{\omega: D(\omega) = \bar{D}} P(\omega|D = \bar{D}) x^R(\bar{D}, q^D; \omega), \\
= \sum_{\omega: D(\omega) = \bar{D}} P(\omega|D = \bar{D}) x^F(\bar{D}, q^D),
\]

where the second inequality is by (4).
Now assume that the result is true for \( k + 1 \). Conditioning on the leftover inventory after \( k \) customers have arrived, we obtain:

\[
\mathbb{E}[x^R(k + 1, q)]|D = k + 1] = \sum_z P[x^R(k, q) = z|D = k + 1]\mathbb{E}[x^R(k + 1, q)|D = k + 1, x^R(k, q) = z],
\]

\[
= \sum_z P[x^R(k, q) = z|D = k + 1] \left( z - \sum_\tau \alpha(\tau)f(\tau, z) \right),
\]

\[
= \sum_z P[x^R(k, q) = z|D = k + 1](z - \rho(z)),
\]

\[
= \mathbb{E}[x^R(k, q)|D = k + 1] - \sum_z P[x^R(k, q) = z|D = k + 1]\rho(z),
\]

\[
= \mathbb{E}[x^R(k, q)|D = k] - \sum_z P[x^R(k, q) = z|D = k]\rho(z).
\] (5)

The third equation is obtained by substituting for \( \rho(z) \). The last one follows from the fact that the expected leftover inventory after \( k \) customers have come is the same whether total demand is \( k \) or \( k + 1 \). On the other hand, we have:

\[
\sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) x^F(k + 1, q^\tau) = \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) \left[ x^F(k, q^\tau) - \rho(x^F(k, q^\tau)) \right],
\]

\[
= \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) \left[ x^F(k, q^\tau) - \rho(x^R(k, q; z)) \right],
\]

\[
= \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) x^F(k, q^\tau) - \sum_{z: D(z) = \overline{D}} \sum_{x^R(k, q; z) = z} P(z|D = \overline{D})\rho(z),
\]

\[
= \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) x^F(k, q^\tau) - \sum_z P[x^R(k, q) = z|D = \overline{D}]\rho(z).
\] (6)

where the second equality is because \( \{ j : x^F_j(k, q^\tau) > 0 \} = \{ j : x^R_j(k, q, z) > 0 \} \) by Claim 1. We can equate (5) and (6) by the induction hypothesis and we get the result for \( k \), which proves Claim 2.

Using Claim 2 for \( k = 0 \), we get

\[
q = \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) q^\tau.
\] (7)

In other words, the weighted average of the starting inventory vectors under FP in our construction is equal to \( q \). Also using Claim 2, we get

\[
\mathbb{E}[y^R(q)] = q - \sum_{k=0}^\tau P[D = k]\mathbb{E}[x^R(k, q)|D = k],
\]

\[
= \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) q^\tau - \sum_{k=0}^\tau P[D = k] \sum_{z: D(z) = \overline{D}} P(z|D = \overline{D}) x^F(k, q^\tau),
\]
\[ = \sum_{\omega} P(\omega|D = \overline{D}) \left[ q^\omega \overline{D} - \sum_{k=0}^{\overline{D}} P[D = k|x^F(k, q^\omega)] \right], \]
\[ = \sum_{\omega} P(\omega|D = \overline{D}) \mathbb{E}[y^F(q^\omega)], \]

where the second equation is by Claim 2 and (7) and the third is obtained by interchanging the two summation signs. In other words, the weighted average of the expected sales in the FP model with starting inventory vectors \( q^\omega \) is equal to the expected sales as the RP model with starting inventory \( q \).

Combining (7) and (8), we have
\[ \mathbb{E}[\Pi^R(q)] = \sum_{j=1}^{n} (u_j + o_j) \mathbb{E}[y_j^R(q)] - \sum_{j=1}^{n} q_j, \]
\[ = \sum_{j=1}^{n} (u_j + o_j) \sum_{\omega} P(\omega|D = \overline{D}) \mathbb{E}[y_j^F(q^\omega)] - \sum_{j=1}^{n} \sum_{\omega} P(\omega|D = \overline{D}) q_j^\omega, \]
\[ = \sum_{\omega} P(\omega|D = \overline{D}) \left[ \sum_{j=1}^{n} (u_j + o_j) \mathbb{E}[y_j^F(q^\omega)] - \sum_{j=1}^{n} q_j^\omega \right], \]
\[ = \sum_{\omega} P(\omega|D = \overline{D}) \mathbb{E}[\Pi^F(q^\omega)]. \]

This, in turn, implies that there exists a \( q^\omega \) such that \( \mathbb{E}[\Pi^F(q^\omega)] \geq \mathbb{E}[\Pi^R(q)] \). So we set \( \tilde{q} = q^\omega \).

Q.E.D.

We illustrate the proof technique for Lemma 1 with a numerical example.

Example 3. (Cont’d from Example 1) Let \( n = 2 \). There are 4 possible types: \( T = \{(1), (2), (1, 2), (2, 1)\} \). All types are equally likely, i.e., \( \alpha(\tau) = 1/4 \) for all \( \tau \in T \). Let \( q = (2, 1) \). Demand is deterministic and \( D = 2 \). Let \( u_1 = 10, u_2 = 1, o_1 = 1, o_2 = 3 \).

Given that \( D = \overline{D} = 2 \) and \( |T| = 4 \), there are 16 possible sample paths, each with probability 1/16. We can put these sample paths into 4 groups: (i) the ones such that both customers buy product 1, (ii) the ones such that the first customer buys product 1 and the second buys product 2, (iii) the ones such that the first customer buys product 2 and the second buys product 1 and (iv) the ones such that the first customer buys product 2 and the second buys nothing.

For example, group (iv) contains the following two sample paths: (a) the two customers are of type (2) (b) the first customer is of type (2,1) and the second customer is of type (2). The combined probability of these two sample paths is 1/8. In group (iv), the retailer runs out of product 2 after the first customer’s visit and ends the period with 2 units of product 1 leftover in the RP model, i.e., \( x^R(D, q, \omega) = (2, 0) \). The starting inventory vector corresponding to the sample paths in group (iv) is \( q^\omega = (13/4, 1/2) \). This is because, when using \( q^\omega \) in the FP model, we also run
out of product 2 after the first customer’s visit and end the period with 2 units of products 1, i.e. \( x^F(D, q^\omega) = (2, 0) \). Also, sales in the FP model with starting inventory \( q^\omega \) are equal to 5/4 units of product 1 and 1/2 units of product 2, i.e., \( y^F(q^\omega) = (5/4, 1/2) \) and expected profit is equal to \((10 + 1)\frac{3}{4} + (1 + 3)\frac{1}{2} = 11\), i.e., \( \Pi^F(q^\omega) = 11 \). In Table 2 we repeat a similar analysis for all four groups of sample paths.

<table>
<thead>
<tr>
<th>( \omega ) such that</th>
<th>Prob.</th>
<th>( q^\omega )</th>
<th>( x^F(2, q^\omega) )</th>
<th>( y^F(q^\omega) )</th>
<th>( \Pi^F(q^\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st and 2nd customers buy 1</td>
<td>1/4</td>
<td>(1, 2)</td>
<td>(0, 1)</td>
<td>(1, 1)</td>
<td>8</td>
</tr>
<tr>
<td>1st customer buys 1, 2nd customer buys 2</td>
<td>1/4</td>
<td>(2, 1)</td>
<td>(1, 0)</td>
<td>(1, 1)</td>
<td>10</td>
</tr>
<tr>
<td>1st customer buys 2, 2nd customer buys 1</td>
<td>3/8</td>
<td>(9/4, 1/2)</td>
<td>(1, 0)</td>
<td>(5/4, 1/2)</td>
<td>12</td>
</tr>
<tr>
<td>1st customer buys 2, 2nd customer buys nothing</td>
<td>1/8</td>
<td>(13/4, 1/2)</td>
<td>(2, 0)</td>
<td>(5/4, 1/2)</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 2 Random inventory vectors \( q^\omega \)'s to be used in the proof of Lemma 1.

With the \( q^\omega \)'s thus constructed for each sample path, we show that their weighted average (using the probability of each sample path as weights) is equal to the starting inventory vector \( q = (2, 1) = \frac{1}{4}(1, 2) + \frac{1}{4}(2, 1) + \frac{3}{8}(9/4, 1/2) + \frac{1}{4}(13/4, 1/2) \). Further, the weighted average of the sales in the FP model using the random inventory vectors \( q^\omega \), matches the expected sales in the RP model when using \( q \): \( E[y^R(q)] = (9/8, 3/4) = \frac{1}{4}(1, 1) + \frac{1}{4}(1, 1) + \frac{3}{8}(5/4, 1/2) + \frac{1}{4}(5/4, 1/2) \). This implies that weighted average of the profit values in the FP model using the \( q^\omega \)'s is equal to the expected profit in the RP using \( q \): \( E[\Pi^R(q)] = 10.375 = \frac{1}{4}8 + \frac{1}{4}10 + \frac{3}{8}12 + \frac{1}{4}11 \).

Therefore, there must exist a value of \( q^\omega \) which is such that its profit under FP is greater than \( E[\Pi^R(q)] \). We can for example take \( q^\omega \) for group (iii), that is, we set \( q = (9/4, 1/2) \). In this case, we have \( \Pi^F(q) = 12 > E[\Pi^R(q)] = 10.375 \). We have found an inventory vector which, under FP, gives a higher expected profit than \( q \) in the RP model.

Proof of Lemma 2

To simplify the notations, we drop the reference to \( q \) and \( \omega \).

The fact that \( k \leq \min_{j=1, \ldots, n} \frac{q_j}{\rho_j} \) implies that no product runs out in the FP model, i.e., \( E[y_j^F|D = k] = y_j^F = k\rho_j \leq q_j \) for \( j = 1, \ldots, n \). Let \( \tilde{y}_j^R \) be the sales under RP if customers do not substitute in the event of a stock-out, i.e., when \( \rho_j = \alpha(j) \) for \( j = 1, \ldots, n \). For every sample path \( \omega \), we have \( y_j^R(\omega) \geq \tilde{y}_j^R(\omega) \), since \( y_j^R \) may include sales from customers who substituted to product \( j \) because another product was out of stock. The number of customers who want to buy product \( j \) in the RP model in the absence of any substitution has a binomial distribution with \( k \) trials and a probability of success of \( \rho_j \). Given that one cannot sell more than \( q_j \) of product \( j \), for \( j = 1, \ldots, n \), expected sales in this case are equal to:

\[
E[\tilde{y}_j^R|D = k] = \sum_{i=0}^{k} \min(i, q_j) C_i^k \rho_j^i (1-\rho_j)^{k-i},
\]
\[ k \rho_j - \sum_{i=0}^{k} (i - q_j) C_i^k \rho_j^i (1 - \rho_j)^{k-i}, \]

\[ y_j F - \sum_{i=0}^{k} (i-q_j) C_i^k \rho_j^i (1 - \rho_j)^{k-i}. \]

Since \( E[y_j^R | D = k] \geq E[y_j^F | D = k] \), we have

\[ (E[y_j^F | D = k] - E[y_j^R | D = k])^+ \leq E[y_j^F | D = k] - E[y_j^R | D = k], \]

\[ = \sum_{i=0}^{k} (i-q_j) C_i^k \rho_j^i (1 - \rho_j)^{k-i}. \]

Let \( \tilde{q}_j = k \rho_j \). Since \( \tilde{q}_j \leq q_j \), we have:

\[ (E[y_j^F | D = k] - E[y_j^R | D = k])^+ \leq \sum_{i=0}^{k} (i - \tilde{q}_j) C_i^k \rho_j^i (1 - \rho_j)^{k-i}. \]

\[ = \sum_{i=0}^{k} (i - \tilde{q}_j) C_i^{[\tilde{q}_j/\rho_j]} (\rho_j^i)^{[\tilde{q}_j/\rho_j] - i}. \]

We now study the behavior of the right-hand side expression as a function of \( \rho_j \), i.e., we consider \( \rho_j^2 < \rho_j^1 \) and compare \( \sum_{i=0}^{[\tilde{q}_j/\rho_j]} (i - \tilde{q}_j) C_i^{[\tilde{q}_j/\rho_j]} (\rho_j^i)^{[\tilde{q}_j/\rho_j] - i} \) and \( \sum_{i=0}^{[\tilde{q}_j/\rho_j]} (i - \tilde{q}_j) C_i^{[\tilde{q}_j/\rho_j]} (\rho_j^i)^{[\tilde{q}_j/\rho_j] - i} \).

First, we show that if \( [\tilde{q}_j/\rho_j^1] = [\tilde{q}_j/\rho_j^2] \equiv n \) then

\[ \sum_{i=0}^{n} (i - \tilde{q}_j) C_i^n (\rho_j^1)^i (1 - \rho_j^1)^{n-i} \geq \sum_{i=0}^{n} (i - \tilde{q}_j) C_i^n (\rho_j^2)^i (1 - \rho_j^2)^{n-i}, \]

that is, the right-hand side of (9) is increasing in \( \rho_j \) between two integer values of \( \tilde{q}_j/\rho_j \). To see this, note that (10) can be rewritten as \( E[X^1 - \tilde{q}_j]^+ \geq E[X^2 - \tilde{q}_j]^+ \), where \( X^1 \sim B(n, \rho_j^1) \) and \( X^2 \sim B(n, \rho_j^2) \). This is true since \( X^1 \) is stochastically larger than \( X^2 \) as a Binomial distribution is stochastically increasing in its probability of success.

Next, suppose that \( [\tilde{q}_j/\rho_j^1] < [\tilde{q}_j/\rho_j^2] \) and in particular, we look at \( \rho_j^1 \) and \( \rho_j^2 \) such that \( \tilde{q}_j/\rho_j^1 \) and \( \tilde{q}_j/\rho_j^2 \) are integer numbers and \( n \equiv \tilde{q}_j/\rho_j^1 = \tilde{q}_j/\rho_j^2 - 1 \). In this case, we show that

\[ \sum_{i=0}^{n} (i - \tilde{q}_j) C_i^n \left( \frac{\tilde{q}_j}{n} \right)^i \left( 1 - \frac{\tilde{q}_j}{n} \right)^{n-i} \leq \sum_{i=0}^{n+1} (i - \tilde{q}_j) C_i^{n+1} \left( \frac{\tilde{q}_j}{n+1} \right)^i \left( 1 - \frac{\tilde{q}_j}{n+1} \right)^{n+1-i}, \]

that is, the right-hand side of (9) is decreasing in \( \rho_j \) if one considers only the points where \( \tilde{q}_j/\rho_j \) is an integer. Note that (11) can be rewritten as showing \( E[X^1 - \tilde{q}_j]^+ \leq E[X^2 - \tilde{q}_j]^+ \), where \( X^1 \sim B(n, \rho_j^1) \) and \( X^2 \sim B(n+1, \rho_j^2) \). To prove this, we show that \( X^2 \) is greater than \( X^1 \) in the convex order, i.e., \( X^1 \leq_{CX} X^2 \) (the random variables \( X^1 \) and \( X^2 \) have the same expected value since...
\( np_j = (n+1)\rho_j = \tilde{q}_j \). This is done by showing that the cdf’s of the two random variables cross only once. Let \( F^1 \) and \( F^2 \) respectively denote the cdf’s of \( X^1 \) and \( X^2 \), i.e., \( F^1(k) = P(X^1 \leq k) \) and \( F^2(k) = P(X^2 \leq k) \). Also, let \( f^1 \) and \( f^2 \) respectively denote the pmf’s of \( X^1 \) and \( X^2 \), i.e., \( f^1(k) = P(X^1 = k) \) and \( f^2(k) = P(X^2 = k) \).

First we show that \( F^1(0) \leq F^2(0) \). We have \( F^1(0) = \left(1 - \frac{\tilde{q}_j}{n}\right)^n \) and \( F^2(0) = \left(1 - \frac{\tilde{q}_j}{n+1}\right)^{n+1} \). To prove the inequality we take the natural logarithm of both expressions:

\[
F^1(0) = \ln \left(1 - \frac{\tilde{q}_j}{n}\right)^n \quad \text{and} \quad F^2(0) = \ln \left(1 - \frac{\tilde{q}_j}{n+1}\right)^{n+1}
\]

and use the Taylor expansion series \( \ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \):

\[
F^1(0) \leq F^2(0) \quad \text{if and only if} \quad -\tilde{q}_j - \frac{\tilde{q}_j^2}{2(n+1)} - \frac{\tilde{q}_j^3}{3(n+1)^2} - \cdots \leq -\tilde{q}_j - \frac{\tilde{q}_j^2}{2n} - \frac{\tilde{q}_j^3}{3n^2} - \cdots
\]

The inequality holds since the \( k \)-th terms on the left hand side of the last expression is smaller or equal than the \( k \)-th term on the right-hand side for \( k = 1, 2, \ldots \).

Since the number of trials is \( n \) for \( X^1 \) and \( n+1 \) for \( X^2 \), we have \( F^1(n) = 1 > F^2(n) \) (and \( F^1(n+1) = 1 = F^2(n+1) \)).

\( F^1(0) \leq F^2(0) \) and \( F^1(n) = 1 > F^2(n) \) together imply that \( F^1 \) and \( F^2 \) have to cross at least once between 0 and \( n \). To prove that they do not cross more than once, we first study the ratios of the pmf’s:

\[
\frac{f^1(i)}{f^2(i)} = \frac{C^n_i \left(\frac{\tilde{q}_j}{n}\right)^i \left(1 - \frac{\tilde{q}_j}{n}\right)^{n-i}}{C^{n+1}_i \left(\frac{\tilde{q}_j}{n+1}\right)^i \left(1 - \frac{\tilde{q}_j}{n+1}\right)^{n+1-i}} = \frac{\left(1 - \frac{\tilde{q}_j}{n}\right)^n}{\left(1 - \frac{\tilde{q}_j}{n+1}\right)^{n+1}} \frac{n+1-i}{n+1} \left(\frac{n+1}{n+1-i}\right)^i.
\]

Taking the natural logarithm of this expression we get:

\[
\ln \left(\frac{\left(1 - \frac{\tilde{q}_j}{n}\right)^n}{\left(1 - \frac{\tilde{q}_j}{n+1}\right)^{n+1}} \right) + \ln(n+1-i) - \ln(n+1) + i \ln \left(1 + \frac{1}{n-\tilde{q}_j}\right).
\]

Differentiating with respect to \( i \), we obtain:

\[
-\frac{1}{n+1-i} + \ln \left(1 + \frac{1}{n-\tilde{q}_j}\right).
\]

Using the Taylor series expansion \( \ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \), we get that \( \ln(1 + x) \geq x \) and therefore

\[
\frac{\partial}{\partial i} \ln \frac{f^1(i)}{f^2(i)} \geq -\frac{1}{n+1-i} + \frac{1}{n-\tilde{q}_j}.
\]
which is greater or equal to zero for \( i \leq \tilde{q}_j \). Hence, \( \frac{f_2(i)}{f_1(i)} \) is increasing for \( i \leq \tilde{q}_j \).

Let us now study the inverse ratio:

\[
\frac{f_2(i)}{f_1(i)} = \frac{C_i^{n+1} \left( \frac{\tilde{q}_j}{n+1} \right)^i \left( 1 - \frac{\tilde{q}_j}{n+1} \right)^{n+1-i}}{C_i^n \left( \frac{\tilde{q}_j}{n} \right)^i \left( 1 - \frac{\tilde{q}_j}{n} \right)^{n-i}} = \left( 1 - \frac{\tilde{q}_j}{n+1} \right)^{n+1} \left( \frac{1}{n+1-i} \right) \left( \frac{n+1}{n+1-i} \right)^{n+1-i}.
\]

Taking the natural logarithm of this expression we get:

\[
\ln \left( \frac{1 - \tilde{q}_j/n}{1 - \tilde{q}_j/(n+1)} \right) + \ln(n+1) - \ln(n+1-i) + i \ln \left( 1 - \frac{1}{n+1-\tilde{q}_j} \right).
\]

Differentiating with respect to \( i \), we obtain:

\[
+ \frac{1}{(n+1-i)} + \ln \left( 1 + \frac{1}{n+1-\tilde{q}_j} \right).
\]

Using the Taylor series expansion \( \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \), we get that \( \ln(1-x) \geq -x \) and therefore

\[
\frac{\partial}{\partial i} \ln \frac{f_2(i)}{f_1(i)} \geq + \frac{1}{(n+1-i)} - \frac{1}{n+1-\tilde{q}_j},
\]

which is greater or equal to zero for \( i \geq \tilde{q}_j \). Hence, \( \frac{f_2(i)}{f_1(i)} \) is increasing for \( i \geq \tilde{q}_j \).

Now assume (contradiction) that \( F^1 \) and \( F^2 \) cross twice. Given that \( F_1(0) < F_2(0) \), we must have \( f_1(i) > f_2(i) \) for some value of \( i \) when or before the two cdfs cross for the first time. In order for the two cdfs to cross again, \( f_1/f_2 \) must be decreasing, hence the second crossing has to be for \( i \geq \tilde{q}_j \). Before or at the second crossing, we must have \( f_1(i) < f_2(i) \). However, after that \( f_2/f_1 \) is increasing so it is not possible for \( F_2 \) to cross \( F_1 \) again. Yet this needs to happen since we know that \( F_1(n) > F_2(n) \). Hence, we have a contradiction.

We have proved that the expression on the right-hand side of (9) is increasing in \( \rho_j \) between two values \( \rho_1 \) and \( \rho_2 \) such that \( \tilde{q}_j/\rho_2 \) then jumps down at \( \rho_3 \) such that \( \tilde{q}_j/\rho_3 = \tilde{q}_j/\rho_2 - 1 \). Also, if one considers the function at the values of \( \rho \) such that \( \tilde{q}_j/\rho_j \) is an integer, then these points are decreasing in \( \rho \). An Example of this behavior is depicted in Figure 6.

It follows that the right-hand side of (9) is maximized when \( \rho_j \to 0 \). And we have:

\[
(E[y_j^k|D = k] - E[y_j^h|D = k])^+ \leq \lim_{\rho_j \to 0} \sum_{i = \lfloor \tilde{q}_j \rfloor}^{\lceil \tilde{q}_j/\rho_j \rceil} (i - \tilde{q}_j)C_i^{\lfloor \tilde{q}_j/\rho_j \rfloor} \rho_j^{-1} \left( 1 - \rho_j \right)^{\lfloor \tilde{q}_j/\rho_j \rfloor - i}.
\]

Consider the sequence \( X \sim B(\lfloor \tilde{q}_j/\rho_j \rfloor , \rho_j) \) for \( \rho_j \to 0 \). Because it is an increasing sequence, it converges to a limit, which is the limit of the subsequence that considers only the integer values
of $[\hat{q}_j / \rho_j]$. The limit of this subsequence has a Normal distribution with mean $\hat{q}_j$ and standard deviation $\sqrt{\hat{q}_j}$. Hence, we have:

$$\left( \mathbb{E}[y^F_j | D = k] - \mathbb{E}[y^R_j | D = k] \right)^+ \leq \int_{k \rho_j}^\infty \frac{1}{\sqrt{\hat{q}_j} \sqrt{2\pi}} \exp \left( -\frac{(z - \hat{q}_j)^2}{2\hat{q}_j} \right) dz,$$

$$= \int_{\sqrt{\hat{q}_j}}^{\infty} \frac{d}{dz} \exp \left( -\frac{(z - \hat{q}_j)^2}{2\hat{q}_j} \right) dz,$$

$$= \frac{\sqrt{\hat{q}_j}}{\sqrt{2\pi}},$$

using $\hat{q}_j = k \rho_j$. Q.E.D.

**Proof of Lemma 3**

Before we provide the proof of Lemma 3, we present another useful result.

**Lemma 4.** Let $\mathbf{q}, \mathbf{q}' \geq 0$ such that $q'_k = q_k - \epsilon \geq 0$ for some $k \in \{1, \ldots, n\}$ and $\epsilon \in \mathbb{N}$, and $q_j = q'_j$ for $j \neq k$, $j = 1, \ldots, n$. For every sample path $\omega$, we have $\sum_{i \neq k} [y^R_i(\mathbf{q}'; \omega) - y^R_i(\mathbf{q}; \omega)] \leq \epsilon$ and $\sum_{i=1}^n [y^R_i(\mathbf{q}'; \omega) - y^R_i(\mathbf{q}; \omega)] \leq 0$.

First, we prove the result for $\epsilon = 1$. We proceed by induction on the number of products with positive inventory in $\mathbf{q}$, denoted by $m$.

First suppose that there is only one product with positive inventory in $\mathbf{q}$, which is product $k$. If the demand for product $k$ is less or equal to $q_k - 1$, the sales of product $k$ are the same whether the starting inventory vector is $\mathbf{q}$ or $\mathbf{q}'$. Otherwise, the sales of product $k$ are higher by one with starting inventory $\mathbf{q}$. Therefore, we have $y^R_k(\mathbf{q}'; \omega) - y^R_k(\mathbf{q}; \omega) \leq 0$ for every $\omega$. Also, we trivially have $\sum_{i \neq k} y^R_i(\mathbf{q}'; \omega) - \sum_{i \neq k} y^R_i(\mathbf{q}; \omega) = 0 \leq 1$. Hence, we have $\sum_{i=1}^n [y^R_i(\mathbf{q}'; \omega) - y^R_i(\mathbf{q}; \omega)] \leq 0$ for every $\omega$. 

![Figure 6](image_url)  
**Figure 6** Right-hand side of (9) with $\hat{q}_j = 10$ as a function of $\rho_j$ for $\rho_j \in [1/3, 1/2]$. 


Now assume that the result is true for $m$ and $\epsilon = 1$. We prove that the result is true for $m+1$ products with positive inventory in $q$ and $\epsilon = 1$. Let system 1 be the system that starts with inventory vector $q$ and system 2 be the one that starts with inventory vector $q'$. Assume $q$ has $m+1$ products have positive inventory. Consider the same sample path of customer arrivals $\omega$ in both systems. There are four cases:

(1) If no product runs out in either system then we have \[ \sum_{i \neq k} [y^R_i(q'; \omega) - y^R_i(q; \omega)] = 0 = \sum_{i=1}^n [y^R_i(q'; \omega) - y^R_i(q; \omega)]. \]

(2) If a product other than $k$ runs out first, the stock-out happens at the same time in both systems and the sales up to that point are the same. After that, only $m$ products have positive inventory therefore, the result is true by the induction hypothesis.

(3) If product $k$ is the first product to run out in system 2 and does not run out in system 1, then \[ \sum_{i \neq k} [y^R_i(q'; \omega) - y^R_i(q; \omega)] = 0 = \sum_{i=1}^n [y^R_i(q'; \omega) - y^R_i(q; \omega)]. \]

(4) If $k$ runs out in both systems, it runs out in system 2 first. After a customer buys the last unit of product $k$ in system 2, system 1 has only 1 unit left of product $k$. Consider the customer who comes to buy product $k$ in system 1. There are two cases. In the first case, this customer does not buy anything (i.e., he does not substitute) in system 2. As a result system 1 has sold one more unit of product $k$ and the sales of products other than $k$ are the same up to then. From then on, the two models are exactly the same, so that \[ \sum_{i \neq k} [y^R_i(q'; \omega) - y^R_i(q; \omega)] = 0, \]
and \[ \sum_{i=1}^n [y^R_i(q'; \omega) - y^R_i(q; \omega)] = -1 \leq 0. \] In the second case, this customer buys product $j \neq k$ (i.e., he substitutes) in system 2. So far system 2 has sold one less unit of product $k$ but one more unit of product $j$ compared to system 1. From then on, both models have only $m$ products with positive inventory, the inventory of products $j \neq i, k$ are the same in both models and system 2 has one less unit of product $j$. By the induction hypothesis, we have that the difference in future sales of products $j \neq k$ between system 2 and system 1 is at most 0. Therefore we have \[ \sum_{i \neq k} [y^R_i(q'; \omega) - y^R_i(q; \omega)] \leq 1 \] and \[ \sum_{i=1}^n [y^R_i(q'; \omega) - y^R_i(q; \omega)] \leq 0. \] This proves that the result is true for all $m$ when $\epsilon = 1$.

Now suppose that $\epsilon = 2$. We construct inventory vector $q''$ as follows. Let $q''_k = q_k - 1 > q''_k = q_k - 2$ and $q''_j = q_j = q'_j$ for $j \neq k$, $j = 1, \ldots, n$. Applying the result for $\epsilon = 1$ to $q$ and $q'$, we have \[ \sum_{i \neq k} [y^R_i(q'''; \omega) - y^R_i(q; \omega)] \leq 1 \] and \[ \sum_{i=1}^n [y^R_i(q'''; \omega) - y^R_i(q; \omega)] \leq 0. \] Then applying the same result to $q'$ and $q''$, we have \[ \sum_{i \neq k} [y^R_i(q'''; \omega) - y^R_i(q'; \omega)] \leq 1 \] and \[ \sum_{i=1}^n [y^R_i(q'''; \omega) - y^R_i(q'; \omega)] \leq 0. \] Combining the two sets of inequalities we get \[ \sum_{i \neq k} [y^R_i(q'''; \omega) - y^R_i(q; \omega)] \leq 2 \] and \[ \sum_{i=1}^n [y^R_i(q'''; \omega) - y^R_i(q; \omega)] \leq 0. \] The same reasoning can be used for any value of $\epsilon \in \mathbb{N}$.

Now we present the proof of Lemma 3.
To simplify the notation, we drop the references to $q$ and $ω$. The proof is by construction and for a given sample path. Start both the FP and RP models with inventory vector $q$. Let $k_1$ be the value of demand at which the FP model runs out for the first time. Without loss of generality assume the FP model runs out of product 1 first.

The sales up until the first stock-out epoch in the FP model are denoted by $y^F_1 = (y^F_{1,1}, y^F_{1,2}, ..., y^F_{1,n})$ and the leftover inventory at the first stock-out epoch in the FP model is denoted by $x^F_1 = (0, x^F_{1,2}, ..., x^F_{1,n})$. Let $y^R_1 = (y^R_{1,1}, y^R_{1,2}, ..., y^R_{1,n})$ denote the sales in the RP model up until the first stock-out epoch and $x^R_1 = (x^R_{1,1}, x^R_{1,2}, ..., x^R_{1,n})$ denote the leftover inventory at the first stock-out epoch in the RP model.

Now we construct a pair of modified systems (one for FP, one for RP) which are similar to the original systems except that we change the leftover inventory vectors at the first stock-out epoch. The variables corresponding these new systems are denote with a $\hat{}$ symbol. The changes are made as follows:

(a) For $j$ such that $x^R_{1,j} > x^F_{1,j}$, the RP model has sold less of product $j$. In this case, let $\hat{x}^R_{1,j} = x^F_{1,j}$, that is, we remove the excess leftover of product $j$ in the RP model. This change may increase future sales of products other than $j$ (because of substitution from product $j$). Let $Z_{k,j}$ for $k \neq j$, $k > 1$ be the increase in future sales of product $k$ due to this change in the leftover inventory of product $j$. By Lemma 4, the total increase $\sum_{k \geq 1}^{k > 1} Z_{k,j}$ cannot be more than $x^R_{1,j} - x^F_{1,j} = y^F_{1,j} - y^R_{1,j}$. Also let $\hat{x}^F_{1,j} = x^F_{1,j}$.

(b) For $j$ such that $x^R_{1,j} < x^F_{1,j}$, the RP model has sold more of product $j$. In this case, let $\hat{x}^F_{1,j} = x^R_{1,j}$, that is, we decrease the leftover inventory of product $j$ in the FP model and count the difference as extra sales. Also let $\hat{x}^R_{1,j} = x^R_{1,j}$.

Sales of product $j$ after the first stock-out epoch are denoted by $y^{F,j}_2$, $y^{R,j}_2$, $\hat{y}^{F,j}_2$, $\hat{y}^{R,j}_2$, respectively in the original FP, original RP, modified FP and modified RP systems.

We have $\hat{y}^{R,j}_{2,j} \geq y^{F,j}_{2,j}$ for $j = 2, ..., n$ because the sale of products satisfying (b) above are increased by $x^F_{1,j} - x^R_{1,j}$ in the modified FP model. Also, the decrease in leftover inventory of these products in turn increases the sales of the other products as the product runs out faster and customer substitute to the other products sooner.

Also, we have $\hat{y}^{R,j}_{2,j} - y^{R,j}_{2,j} \leq \sum_{k \neq j}^k Z_{j,k}$ for $j = 1, ..., n$. And therefore, $\sum_{j=1}^n (\hat{y}^{R,j}_{2,j} - y^{R,j}_{2,j})^+ \leq \sum_{j=1}^n \sum_{k \neq j}^k Z_{j,k} \leq \sum_{j=1}^n (y^{F,j}_{1,j} - y^{R,j}_{1,j})^+$.

Using these results we obtain:

$$\sum_{j=1}^n (y^{F,j}_j - y^{R,j}_j)^+ \leq \sum_{j=1}^n (y^{F,j}_{1,j} - y^{R,j}_{1,j})^+ + \sum_{j=2}^n (y^{F,j}_{2,j} - y^{R,j}_{2,j})^+,$$
be obtained by sorting the values of \( \rho_{i,j} \) where \( j \) is the proportion of these customers wanting to buy product \( j \), \( i \) is the number of customers that come between the \((i-1)\)-th and \( i \)-th stock-out epochs in the modified FP models and \( \rho_{i,j} \) is the proportion of these customers wanting to buy product \( j \). Hence, we have:

\[
\sum_{j=1}^{n} \left( \mathbb{E}[y_{i,j}^{F}] - \mathbb{E}[y_{i,j}^{R}] \right) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} \sum_{i=1}^{j} \sqrt{k_{i}\rho_{i,j}}.
\]

By Lemma 2, we have \( (\mathbb{E}[y_{i,j}^{F}] - \mathbb{E}[y_{i,j}^{R}])^{+} \leq \frac{\sqrt{k_{i}\rho_{i,j}}}{\sqrt{\pi}} \) for \( i = 1, \ldots, n, \) \( j = i, \ldots, n \), where \( k_{i} \) is number of customers that come between the \((i-1)\)-th and \( i \)-th stock-out epochs in the modified FP models and \( \rho_{i,j} \) is the proportion of these customers wanting to buy product \( j \). Hence, we have:

\[
\sum_{j=1}^{n} \left( \mathbb{E}[y_{i,j}^{F}] - \mathbb{E}[y_{i,j}^{R}] \right) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} \sum_{i=1}^{j} \sqrt{k_{i}\rho_{i,j}}.
\]

For \( j = 1, \ldots, n \), the vector \( (k_{1}\rho_{1,j}, \ldots, k_{j}\rho_{j,j}) \) majorizes the vector \( \left( \sum_{i=1}^{j} k_{i}\rho_{i,j}, \ldots, \sum_{i=1}^{j} k_{i}\rho_{i,j} \right) \) where majorization is defined in Chapter 3 of Marshall and Olkin (1979). Hence, we can use a standard result on majorization from Marshall and Olkin (1979) to get \( \sum_{i=1}^{j} k_{i}\rho_{i,j} \leq j \sqrt{\sum_{i=1}^{j} k_{i}\rho_{i,j}} \) for \( j = 1, \ldots, n \). Also, since \( q_{j} \geq \sum_{i=1}^{j} k_{i}\rho_{i,j} \), we get \( \sqrt{\sum_{i=1}^{j} k_{i}\rho_{i,j}} \leq j \sqrt{q_{j}} \) for \( j = 1, \ldots, n \). Hence,

\[
\sum_{j=1}^{n} \left( \mathbb{E}[y_{i,j}^{F}] - \mathbb{E}[y_{i,j}^{R}] \right) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} j \sqrt{q_{j}} = \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{jq_{j}}.
\]

In this expression \( j \) is the \( j \)-th product to run out. An upper bound on the right-hand side can be obtained by sorting the values of \( q_{j} \) in increasing order, that is,

\[
\sum_{j=1}^{n} \left( \mathbb{E}[y_{i,j}^{F}] - \mathbb{E}[y_{i,j}^{R}] \right) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} \sqrt{jq_{[j]}},
\]

where \( q_{[j]} \) is the \( j \)-th greatest value in \( q_{1}, \ldots, q_{n} \).
For a fixed value of \( Q = \sum_{j=1}^{n} q_j \), the right-hand side of the previous expression is maximized when \( q_j = \frac{2jQ}{n(n+1)} \) for \( j = 1, \ldots, n \), so we have:

\[
\sum_{j=1}^{n} (E[y_j^F] - E[y_j^R])^+ \leq \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{j=1}^{n} j \sqrt{\frac{2Q}{n(n+1)}},
\]

\[
= \frac{\sqrt{n(n+1)}}{\sqrt{\pi}} \sqrt{Q}.
\]

Hence we get:

\[
\frac{\sum_{j=1}^{n} (E[y_j^F] - E[y_j^R])^+}{Q} \leq \frac{\sqrt{n(n+1)}}{\sqrt{\pi} \sqrt{Q}}.
\]

**Proof of Proposition 3**

We have

\[
E[\Pi^F(q^*F)] - E[\Pi^R(q^*F)] = \sum_{j=1}^{n} (u_j + o_j) (E[y_j^F(q^*F)] - E[y_j^R(q^*F)]),
\]

\[
\leq \sum_{j=1}^{n} (u_j + o_j) (E[y_j^F(q^*F)] - E[y_j^R(q^*F)])^+,
\]

\[
\leq \left( \max_{j=1,\ldots,n} (u_j + o_j) \right) \left( \max_{(i_1,\ldots,i_n)} \sum_{j=1}^{n} 2j\gamma \left( \frac{q_{ij}^F}{j} \right) \right),
\]

where \((i_1,\ldots,i_n)\) is a permutation of \((1,\ldots,n)\).